

## Lecture 8, Th., Feb. 8, 2007

**Reading homework:** Page 393-399 of Mathematical Models in Biology by Leah Edelstein. <http://ec-securehost.com/SIAM/CL46.html>.

We provided a detailed derivation of the one dimensional conservation equation and sketched the derivation of the three dimensional conservation equation.

**1. Derivation of the conservation equation.** The conservation equation in its various forms provide a basic and effective platform to describe changes in spatial distributions. Most of the PDEs we will encounter in the context of mathematical biology and medicine are in essence built on such sound platform.

We present below a detailed derivation of the one dimensional conservation law in a general form. We let  $x$  represent the distance along a tube with varying cross section area  $A(x, t)$  from some location. We are interested in tracking the concentration  $c(x, t)$  of some particles (cells, species, etc) in the tube. We will pay close attention on the tube section from  $x$  to  $x + \Delta x$ . The changes in the concentration can be accounted by two possible effects: (1) flow of particles into and out of the interval  $(x, x + \Delta x)$ , and (2) processes that import particles or export particles locally. The conservation/balance equation can be written either in terms of mass or number of particles. We simply choose the latter description. The conservation equation states that the rate of change of particles in the section  $(x, x + \Delta x)$  is equal to the rate of particle entry into that section minus the rate of particle departure in the same section, plus the rate of net local change (denoted by  $\sigma(x, t)$ ) in that section due to other processes.

For convenience and clarity, we clearly define the following quantities:

- $c(x, t)$  = concentration of particles (number per unit volume) at  $(x, t)$ ,
- $\mathbf{J}(x, t)$  = flux of particles at  $(x, t)$  = number of particles crossing a unit area at  $x$  in the positive direction per unit time,
- $\sigma(x, t)$  = sink/source density = number of particles created or eliminated per unit volume at  $(x, t)$ .

It is assumed that the only flux that changes the total population is that entering or leaving through the cross sections at  $x$  and  $x + \Delta x$ , namely,  $\mathbf{J}(x, t)$  and  $\mathbf{J}(x + \Delta x, t)$ .

By the definition of the concentration  $c(x, t)$ , we have

$$\int_{x_1}^{x_2} c(x, t)A(x, t)dx = \text{number of particles in } (x_1, x_2) \text{ at time } t.$$

Similarly, the definition of the source density  $\sigma(x, t)$  implies that

$$\int_{x_1}^{x_2} \sigma(x, t)A(x, t)dx = \text{net rate of particle production in } (x_1, x_2) \text{ at time } t.$$

The conservation equation can then be written in integral form (often called the weak form), as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{x_0}^{x_0+\Delta x} c(x, t)A(x, t)dx &= \mathbf{J}(x_0, t)A(x_0, t) - \mathbf{J}(x_0 + \Delta x, t)A(x_0 + \Delta x, t) \\ &+ \int_{x_0}^{x_0+\Delta x} \sigma(x, t)A(x, t)dx. \end{aligned}$$

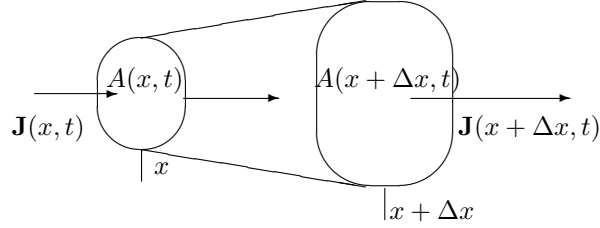


FIG. 1.1. Illustration for the derivation of the one dimensional conservation equation.  $A(x, t)$  is the varying cross area at  $(x, t)$ ,  $\mathbf{J}(x, t)$  is the flux of particles at  $(x, t)$ .

An integral mean value theorem allows one to conclude that there are  $x_1, x_2 \in (x_0, x_0 + \Delta x)$ , such that the following is true:

$$\begin{aligned} \frac{\partial}{\partial t} [c(x_1, t)A(x_1, t)]\Delta x &= \mathbf{J}(x_0, t)A(x_0, t) - \mathbf{J}(x_0 + \Delta x, t)A(x_0 + \Delta x, t) \\ &\quad + [\sigma(x_2, t)A(x_2, t)]\Delta x. \end{aligned}$$

Observe that  $x_1, x_2 \rightarrow x_0$  when  $\Delta x \rightarrow 0$ . By dividing through by  $\Delta x$  and letting  $\Delta x \rightarrow 0$ , we obtain

$$(1.1) \quad \frac{\partial}{\partial t} c(x_0, t)A(x_0, t) = -\frac{\partial}{\partial x} \mathbf{J}(x_0, t)A(x_0, t) + \sigma(x_0, t)A(x_0, t).$$

Derivations for higher dimensional conservation equation is largely similar to the one dimensional case, except care must be exercised when computing the net flux crossing through a rectangular region (for two dimension) or a box (for three dimension). Three dimensional conservation equation takes the form of

$$(1.2) \quad \frac{\partial}{\partial t} c(x, y, t) = -\nabla \cdot \mathbf{J}(x, y, t) + \sigma(x, y, t),$$

where  $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ .

**Fick's law** states that the flux due to random motion is approximately proportional to the local gradient in the particle concentration. That is

$$(1.3) \quad \mathbf{J} = -D\nabla c.$$

Hence, a three dimensional reaction diffusion equation takes the form of

$$(1.4) \quad \frac{\partial}{\partial t} c(x, y, t) = -D\Delta c(x, y, t) + \sigma(x, y, t),$$

where  $\Delta = \nabla \cdot \nabla = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$ .

## EXERCISES

Due on Th., Feb. 22

- (2). Give a detailed derivation of the two dimensional conservation equation.

$$(1.5) \quad \frac{\partial}{\partial t} c(x, y, t) = -\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \mathbf{J}(x, y, t) + \sigma(x, y, t).$$

- (3). The cross-sectional area of the small intestine varies periodically in space and time due to peristaltic motion of the gut muscles. Suppose that at position  $x$  (where  $x$  = length along the small intestine) the area can be described by

$$A(x, t) = a[2 + \cos(x - vt)]/2,$$

where  $v$  is a constant.

1. Write an equation of balance for  $e(x, t)$ , the concentration of digested material at location  $x$ .
2. Suppose there is a constant flux, of material throughout the intestine from the stomach [that is,  $\mathbf{J}(x, t) = 1$ ] and that material is absorbed from the gut into the bloodstream at a rate proportional to its concentration for every unit area of intestinal wall. Formulate the appropriate balance equation.
3. Show that even if  $\mathbf{J}(x, t) = 0$  and  $\sigma(x, t) = 0$ , the concentration  $c(x, t)$  appears to change.