Maximal Coactions

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Abstract:

Efforts to generalize Mansfield's Imprimitivity Theorem have encountered difficulty, in part because there are several types of C^*-dynamical systems, and several types of crossed products. In this paper we resolve some of these problems by introducing yet another type of system (*maximal coactions*), and another, *intermediate*, type of crossed product. We show that duality theorems (a special kind of imprimitivity theorem) hold for maximal coactions and *full* crossed products, and in general for intermediate crossed products. These results parallel a known result for minimal coactions and reduced crossed products. They organize the theory of coactions, crossed products, and duality into a very appealing and symmetric framework, and are expected to lead to very general versions of Mansfield's theorem. Indeed, the first and third authors have already shown how this works in the context of discrete coactions.

This is joint work with Siegfried Echterhoff and John Quigg.

A Duality Theorem for Coactions (A, G, δ):

Theorem 1 (K, Quigg [2])

$$A \times_\delta G \times_{\widehat{\delta},r} G \cong (A/\ker j_A) \otimes \mathcal{K}(L^2(G))$$

Definition 2 *A coaction* (A, G, δ) *is* normal *if* $j_A\colon A \to M(A \times_\delta G)$ *is injective.*

Thus, for normal coactions δ:

$$A \times_\delta G \times_{\widehat{\delta},r} G \cong A \otimes \mathcal{K}(L^2(G))$$

Proposition 3 (Raeburn [6]; Quigg [5]) *Every coaction* (A, G, δ) *has a* normalization*: a normal coaction* (A^n, G, δ^n) *and a* $\delta - \delta^n$ *equivariant surjection* $\psi\colon A \to A^n$ *such that*

$$\psi \times G\colon A \times_\delta G \to A^n \times_{\delta^n} G$$

is an isomorphism.

Proposition 4 *For any coaction (A, G, δ), there exists an isomorphism Υ such that TFDC:*

$$
\begin{array}{ccc}
A \times_\delta G \times_{\widehat{\delta}} G & \xrightarrow{\ \Phi\ } & A \otimes \mathcal{K}(L^2(G)) \\
\Big\downarrow{\scriptstyle \Lambda} & & \Big\downarrow{\scriptstyle \psi \otimes \mathrm{id}} \\
A \times_\delta G \times_{\widehat{\delta}, r} G & \xrightarrow[\ \Upsilon\ ]{} & A^n \otimes \mathcal{K}(L^2(G))
\end{array}
$$

Λ: *regular representation*
Φ: *canonical surjection*

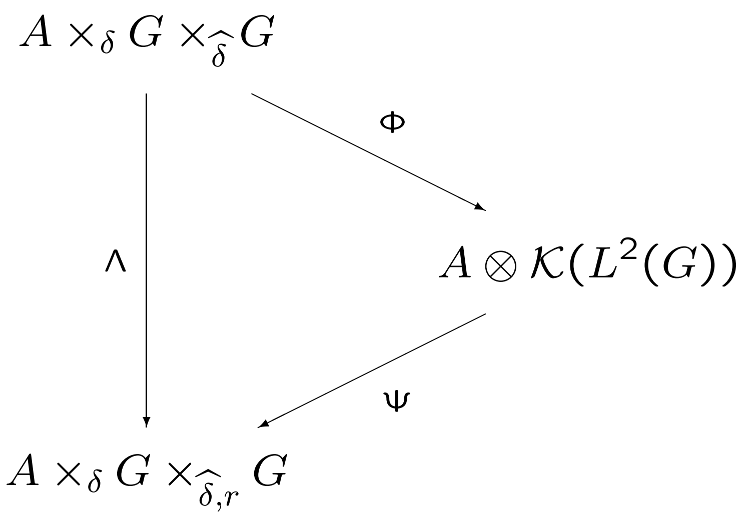
$$\Phi = ((\mathrm{id}_A \otimes \lambda) \circ \delta) \times (1 \otimes M) \times (1 \otimes \rho)$$

Proof. Since Λ, Φ, $\psi \otimes \mathrm{id}$ are all surjections, it suffices to verify that

$$\ker \Lambda = \ker(\psi \otimes \mathrm{id}) \circ \Phi.$$

□

Surjections:



Corollary 5 *δ is normal if and only if the surjection*

$$\Psi = \Upsilon^{-1} \circ (\psi \otimes \mathrm{id})$$

is an isomorphism.

Definition 6 *A coaction (A, G, δ) is* maximal *if the canonical surjection*

$$\Phi\colon A \times_\delta G \times_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$$

is an isomorphism.

Proposition 7 *For any action* (B, G, β)*, the dual coaction*

$$\widehat{\beta} = (i_B \otimes 1) \times (i_G \otimes u)$$

of G *on* $B \times_\beta G$ *is maximal.*

Compare with the "Full Mansfield" (cf. [4]) for dual coactions of Echterhoff, K, Raeburn [1]: for any closed normal subgroup N of G,

$$(B \times_\beta G) \times_{\widehat{\beta}} G \times_{\widehat{\widehat{\beta}}|} N \sim (B \times_\beta G) \times_{\widehat{\beta}|} G/N.$$

Taking $N = G$ we have a "Full Katayama" (cf. [3]):

$$(B \times_\beta G) \times_{\widehat{\beta}} G \times_{\widehat{\widehat{\beta}}} G \sim (B \times_\beta G).$$

Proposition 8 *If* (A, G, δ) *and* (B, G, ϵ) *are Morita equivalent coactions, then* δ *is maximal if and only if* ϵ *is maximal.*

Proof. Show that

$$\ker \Phi_A \leftrightarrow \ker \Phi_B$$

under the Rieffel correspondence between ideals set up by

$$A \times_\delta G \times_{\hat{\delta}} G \sim B \times_\epsilon G \times_{\hat{\epsilon}} G.$$

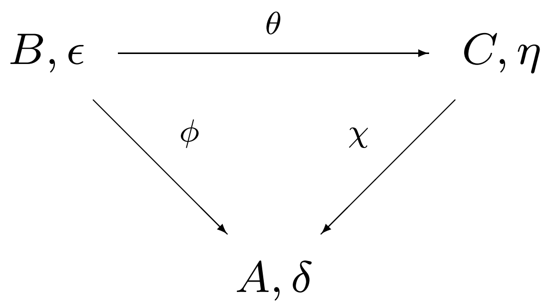
□

Definition 9 *Let* (A, G, δ) *be a coaction. A maximal coaction* (B, G, ϵ) *is a* maximalization *of* δ *if there exists an* $\epsilon - \delta$ *equivariant surjection* $\phi: B \to A$ *such that*

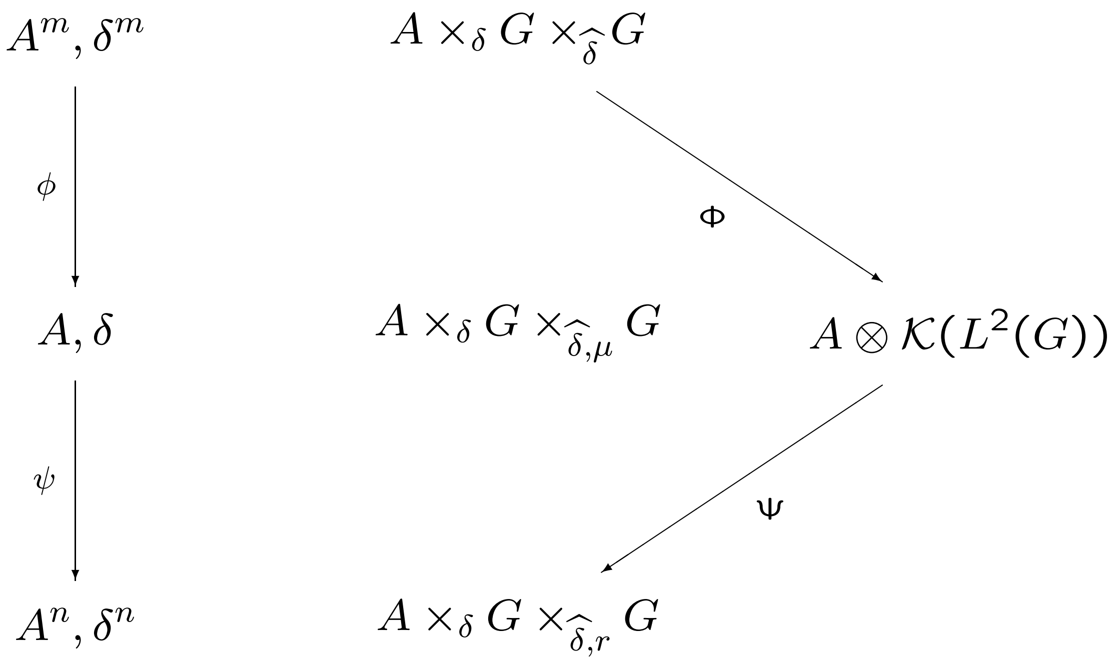
$$\psi \times G: B \times_{\epsilon} G \to A \times_{\delta} G$$

is an isomorphism.

Theorem 10 *Every coaction has a maximalization. If* (B, G, ϵ) *and* (C, G, η) *are maximalizations of* (A, G, δ) *with equivariant surjections* $\phi: B \to A$ *and* $\chi: C \to A$*, then there exists an* $\epsilon - \eta$ *equivariant isomorphism* $\theta: B \to A$ *such that* $\chi \circ \theta = \phi$*.*



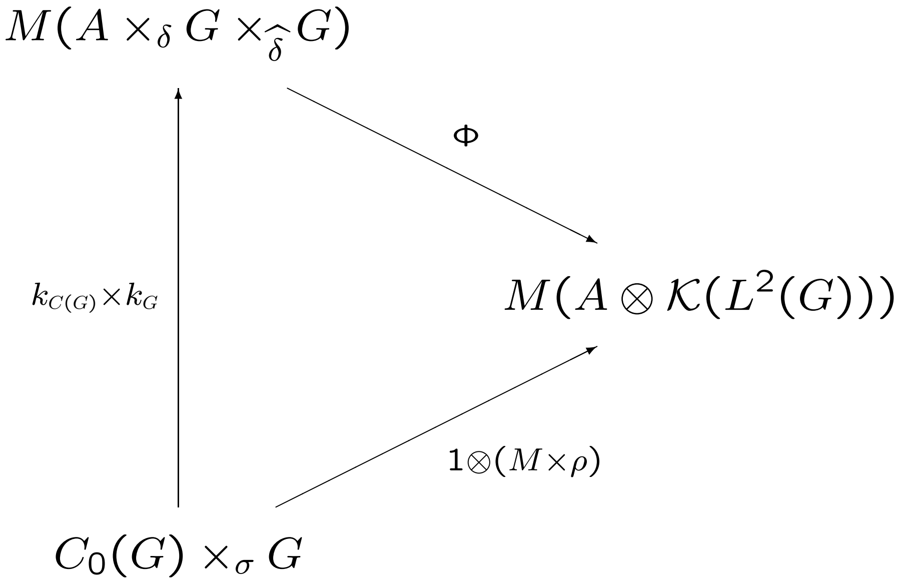
Intermediate Crossed Products:



Proof. If (A^m, G, δ^m) were a maximalization of (A, G, δ), we'd have

$$A \times_\delta G \times_{\widehat{\delta}} G \cong A^m \times_{\delta^m} G \times_{\widehat{\delta^m}} G \cong A^m \otimes \mathcal{K}(L^2(G)).$$

Lemma 11 *For any coaction* (A, G, δ)*, TFDC:*



Put

$$A^m = p(A \times_\delta G \times_{\widehat{\delta}} G)p,$$

where P is a rank-one projection in $\mathcal{K}$ and

$$p = (k_{C(G)} \times k_G) \circ (M \times \rho)^{-1}(P).$$

Lemma 12 *The dual coaction* $\widehat{\widehat{\delta}}$ *of* G *on* $A \times_\delta G \times_{\widehat{\delta}} G$ *is exterior equivalent to a coaction* $\tilde{\delta}$ *for which* p *is invariant:*

$$\tilde{\delta}(p) = p \otimes 1.$$

In particular,

$$\delta^m = \tilde{\delta}|_{A^m}$$

is a (nondegenerate) coaction of G on A^m, which is maximal because it is Morita equivalent to a dual coaction.

Lemma 13 *The canonical surjection* Φ *is* $\tilde{\delta} - \delta \otimes_* \mathrm{id}$ *equivariant.*

Thus,

$$\phi = \Phi|_{A^m} : A^m \to A$$

is a $\delta^m - \delta$ equivariant surjection.

Thus far,

$$\begin{array}{ccc}
A \times_\delta G \times_{\widehat{\delta}} G, \tilde{\delta} & \xrightarrow{\Phi} & A \otimes \mathcal{K}, \delta \otimes_* \mathrm{id} \\
\Big\uparrow{\scriptstyle p} & & \Big\uparrow{\scriptstyle 1 \otimes P} \\
A^m, \delta^m & \xrightarrow{\phi} & A, \delta
\end{array}$$

so

$$\begin{array}{ccc}
A \times_\delta G \times_{\widehat{\delta}} G \times_{\tilde{\delta}} G & \xrightarrow{\Phi \times G} & (A \otimes \mathcal{K}) \times_{\delta \otimes_* \mathrm{id}} G \\
\Big\uparrow{\scriptstyle j(p)} & & \Big\uparrow{\scriptstyle j(1 \otimes P)} \\
A^m \times_{\delta^m} G & \xrightarrow{\phi \times G} & A \times_\delta G.
\end{array}$$

So $\phi \times G$ is an isomorphism if $\Phi \times G$ is.

Recall:

$$\begin{array}{ccc}
A \times_\delta G \times_{\widehat{\delta}} G, \tilde{\delta} & \xrightarrow{\Phi} & A \otimes \mathcal{K}, \delta \otimes_* \mathsf{id} \\
\Big\downarrow{\scriptstyle \Lambda} & & \Big\downarrow{\scriptstyle \psi \otimes \mathsf{id}} \\
A \times_\delta G \times_{\widehat{\delta},r} G, (\tilde{\delta})^n & \xrightarrow{\cong} & A^n \otimes \mathcal{K}, \delta^n \otimes_* \mathsf{id}
\end{array}$$

Since the vertical arrows are normalizations,

$$\begin{array}{ccc}
A \times_\delta G \times_{\widehat{\delta}} G \times_{\tilde{\delta}} G & \xrightarrow{\Phi \times G} & (A \otimes \mathcal{K}) \times_{\delta \otimes_* \mathsf{id}} G \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
A \times_\delta G \times_{\widehat{\delta},r} G \times_{(\tilde{\delta})^n} G & \xrightarrow{\cong} & (A^n \otimes \mathcal{K}) \times_{\delta^n \otimes_* \mathsf{id}} G
\end{array}$$

$\square$

References:

[1] S. Echterhoff, S. Kaliszewski, and I. Raeburn, *Crossed products by dual coactions of groups and homogeneous spaces*, J. Operator Theory **39** (1998), 151–176.

[2] S. Kaliszewski and J. Quigg, *Imprimitivity for C^*-coactions of non-amenable groups*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 101–118.

[3] Y. Katayama, *Takesaki's duality for a non-degenerate co-action*, Math. Scand. **55** (1985), 141–151.

[4] K. Mansfield, *Induced representations of crossed products by coactions*, J. Funct. Anal. **97** (1991), 112–161.

[5] J. Quigg, *Full and reduced C^*-coactions*, Math. Proc. Camb. Phil. Soc. **116** (1995), 435–450.

[6] I. Raeburn, *On crossed products by coactions and their representation theory*, Proc. London Math. Soc. **3** (1992), 625–652.