ARIZONA STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Chapter 16 Vector Calculus

Section 16.1 Direction Fields

Suppose we are given a first order differential equation of the form $dy/dx = y' = F(x,y)$ where $F(x,y)$ is some expression in x, y. Separable equations are the special case in which $F(x,y)$ can be factored as a function of x times a function of y. In some cases it is hard or impossible to find a formula for the solution, but still we can visualize the solution curves by means of a **direction field**. If a solution curve passes through a point (x, y) then its slope at that point is $dy/dx = y' = F(x,y)$. If we draw short line segments with slope $dy/dx = y' = F(x,y)$ at several points (x, y), the result is called a direction field or slope field. These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

Definition: Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x, y) in D a two-dimensional vector $\mathrm{F}(x, y)$.

Definition: Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x, y, z) in D a three-dimensional vector $\mathrm{F}(x, y, z)$.

Examples:

1. Sketch the direction field for $y' = F(x,y) = x^2 + y^2 - 1$ and sketch the solution curve that passes through the origin.

Solution: We calculate the following points:

x	-2	-1	0	2	-2	-1	0	1	1	2	--	--
y	0	0	0	0	1	1	1	1	0	1	--	--
y'	3	0	-1	3	4	1	0	1	0	4	--	--

2. Sketch the vector field $F(x, y) = (x - y)i + xj$ by drawing a diagram.

x	0	0	1	-1	0	--	--	--	--
y	0	1	0	0	-1	--	--	--	--
F	<0,0>	<-1,0>	<1,1>	<-1,-1>	<1,0>	--	--	--	--

3. Sketch the vector field $F(x, y) = yj$ by drawing a diagram.
4. Sketch the vector field $F(x, y) = \dfrac{yi + xj}{\sqrt{x^2 + y^2}}$ by drawing a diagram.

5. Sketch the vector field $F(x,y)=\dfrac{yi-xj}{\sqrt{x^2+y^2}}$ by drawing a diagram.
6. Sketch the vector field $F(x,y)=zj$ by drawing a diagram.
7. Sketch the direction field of the scalar function $f(x,y)=\sin(x+y)$ by drawing a diagram. Remember that $F(x,y)=\nabla f$
8. Find the gradient field of $f(x,y)=\ln(x+2y)$ and sketch it. Remember to find $F(x,y)=\nabla f$
9. Sketch the gradient field of $\phi(x,y)=x+y$

Section 16.2 Line Integrals

The first goal of this section is to define what it means to integrate a function along a curve. To motivate the definition we will consider the problem of finding the mass of a thin wire whose linear density function (mass per unit length) is known. We assure that we can model the wire by a smooth curve C between two points P and Q in 3-space. Given any point (x,y,z) on C we let $f(x,y,z)$ denote the corresponding value of the density function. To compute mass of the wire divide C into n small sections using succession of distinct partition points $P=p_0,p_1,p_2,\cdots,p_n=Q$ and $\triangle M_k\approx f(x_k^*,y_k^*,z_k^*)\triangle S_k$, where $\triangle M_k$ be the mass of the kth section and $\triangle S_k$ be the length between p_k and p_{k-1}. We then find $M=\int_C f(x,y,z)dS$. And in 2-D $M=\int_C f(x,y)dS$

If C is a smooth curve parameterized by $r(t)=x(t)i+y(t)j,\ a\le t\le b$ then

$$\int_C f(x,y)dS=\int_a^b f(x(t),y(t))\left|r'(t)\right|dt$$

Examples:

1. Evaluate the line integral $\int_C f(x,y)dS, r(t)=ti+2tj,\ 0\le t\le 1$

Solution: $\int_C f(x,y)dS=\int_a^b f(x(t),y(t))\left|r'(t)\right|dt=\int_0^1(1+4t^3)\sqrt{5}dt=2\sqrt{5}$

2. Evaluate the line integral $\int_C (xy+z^3)dS$ from (1, 0, 0) to $(-1,0,\pi)$ along the helix C that is represented by $x=\cos t, y=\sin t, z=t, 0\le t\le\pi$, and $dS=\sqrt{(dx/dt)^2+(dy/dt)^2+1}\,dt$

Solution: $\int_C (xy+z^3)dS=\int_0^\pi(\cos t\sin t+t^3)\sqrt{2}dt=\sqrt{2}\pi^4/4$

3. Evaluate the line integral $\int_C (2+x^2y)dS$ where C is the upper half of the unit circle $x^2+y^2=1$.

Solution: $x=\cos t, y=\sin t, 0\le t\le \pi$, and $dS=\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=dt$

$$\int_C (2+x^2y)dS=\int_0^{\pi}(2+\cos^2 t\sin t)dt=2\pi+2/3$$

4. Evaluate the line integral $\int_C 5xdS$ where C consists of the arc C_1 of the parabola $y=2x^2$ from (0,0) to (1,2) followed by the vertical line segment C_2 from (1,2) to (1,4).

Solution: On C_1 curve $y=2x^2, 0\le x\le 1$

$$\int_{C_1} 5xdS=\int_0^1 5x\sqrt{1+(dy/dx)^2}\,dx=\int_0^1 5x\sqrt{1+16x^2}\,dx=5/48(17\sqrt{17}-1)$$

And On C_2 curve $x=1, 2\le y\le 4$

$$\int_{C_1} 5xdS=\int_2^4 5x\sqrt{1+(dx/dy)^2}\,dy=\int_0^1 5x\sqrt{1+0}\,dy=10$$

Thus $\int_C 5xdS=10+5/48(17\sqrt{17}-1)$

5. A wire takes the shape of the semicircle $x^2+y^2=1, y\ge 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution: $x=\cos t, y=\sin t, 0\le t\le \pi$, and $dS=\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=dt$

For mass $m=\int_C \rho(x,y)dS=\int_0^{\pi} k(1-y)dS=k(\pi-2)$

And $\overline{x}=\frac{1}{m}\int_C x\rho(x,y)dS=\int_0^{\pi} kx(1-y)dS=0$ and

$$\overline{y}=\frac{1}{m}\int_C y\rho(x,y)dS=\int_0^{\pi} ky(1-y)dS=\frac{4-\pi}{2(\pi-2)}$$

Homework Problems:

6. Evaluate the line integral a) $\int_C xe^y dx$ where C is the arc of the curve $x=e^y$ from $(1,0)$ to $(e,1)$

Solution: $\int_c xe^y dx=\int_1^e x^2 dx=1/3(e^3-1)$

8. Evaluate the line integral a) $\int_C \sin x dx + \cos y dy$ where C is the arc consisting the top half of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ and the line segment from $(-1,0)$ to $(-2,3)$

Solution: $\int_C \sin x dx + \cos y dy = \int_{C_1} \sin x dx + \cos y dy + \int_{C_2} \sin x dx + \cos y dy$

$$\int_C \sin x dx + \cos y dy = \int_0^{\pi} \sin(\cos t)(-\sin t)dt + \cos(\sin t)\cos t dt$$

$$+ \int_{-1}^{-2} \sin x dx + \cos(3x+3)(-3)dx$$

$$= \cos 1 - \cos 2 + \sin 3$$

14. Evaluate $\int_C zdx + xdy + ydz,\ C: x = t^2, y = t^3, z = t^2, 0 \le t \le 1$

Solution: $\int_C zdx + xdy + ydz = \int_0^1 t^2(2t)dt + t^2(3t^2)dt + t^3(2t)dt = 3/2$

22. $\int_C F \cdot dr$; $F(x, y, z) = zi + yj - xk, r(t) = ti + \sin tj + \cos tk, 0 \le t \le \pi$

Solution: $\int_C F \cdot dr = \int_0^{\pi} F(r(t) \cdot r'(t)dt = \pi$, where

$F = zi + yj - xk = i\cos t + j\sin t - kt, r'(t) = <1, -\cos t, -\sin t>$

32. Find the mass and center of mass of a thin wire in the shape of a quarter circle $x^2 + y^2 = r^2$, $x \ge 0, y \ge 0$, if the density function is $\rho(x, y) = x + y$

Solution: $x = r\cos t, y = r\sin t, 0 \le t \le \pi/2, ds = \sqrt{\dot{x}^2 + \dot{y}^2}dt = rdt$

$$m = \int_C (x+y)ds = \int_0^{\pi/2} r^2(\sin t + \cos t)dt = 2r^2$$

$$\bar{x} = \frac{1}{m}\int_C x(x+y)ds = \frac{r(\pi+2)}{8},\quad \bar{y} = \frac{1}{m}\int_C y(x+y)ds = \frac{r(\pi+2)}{8}$$

Section 16.3 Fundamental Theorem for Line Integrals

Fundamental Theorem for line integrals:

1. $\int_a^b F'(x)dx = F(b) - F(a)$

2. C is a smooth curve given by $r(t), a \le t \le b,\ \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$

3. In general $\int_{C_1} \nabla f \cdot dr \neq \int_{C_2} \nabla f \cdot dr$. But when ∇f is continuous then $\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr$

4. $F = Pi + Qj$ on an open simply connected region D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then F is conservative.

Examples:

1. Given that $F(x, y, z) = y^2 i + (2xy + e^{3z}) j + 3ye^{3z} k$, find *f* such that $F = \nabla f$

   Answer: $f(x, y, z) = xy^2 + ye^{3z} + C$

2.

Homework problems:

Given $F = Pi + Qj$ on an open simply connected region D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then F is conservative.

4. Is $F(x, y) = (x^2 + 4xy)i + (4xy - y^3) j$ conservative vector field? If it is, find *f* such that $F = \nabla f$

Solution: $\frac{\partial P}{\partial y} = 4x, \frac{\partial Q}{\partial x} = 4y$, not conservative

8. Is $F(x, y) = (1 + 2xy + \ln x)i + (x^2) j$ conservative vector field? If it is, find *f* such that $F = \nabla f$

Solution: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$. F is conservative.

Now $F = \nabla f = <1 + 2xy + \ln x, x^2> = <f_x, f_y>$

Now you try to find that $f(x, y) = x^2 y + x \ln x + K$, where K is a constant.

14. $F(x, y) = \frac{y^2}{1 + x^2} i + j 2y \arctan x,\ C : r(t) = t^2 i 2tj, 0 \le t \le 1$

Solution: a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{2y}{1 + x^2}$. F is conservative.

Now find that $f(x, y) = y^2 \arctan x + K$

b)

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$
$$= f(r(1)) - f(r(0)) = f(1,2) - f(0,0) = \pi$$

20. Show that the line integral $\int_C (1-ye^{-x})dx + e^{-x}dy$ is independent where C is the path from (0,1) to (1,2). And evaluate the integral.

Solution: We know that the line integral of any conservative field is independent of the path. Since we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -e^{-x}$, the integral is independent.

Now $\nabla f = < 1-ye^{-x}, e^{-x} >$, you can now find that $f(x,y) = x + ye^{-x}$

$\int_C (1-ye^{-x})dx + e^{-x}dy = \int_C F \cdot dr = f(1,2) - f(0,1) = 2/e$

22. Find work done by the force field $F = \frac{y^2}{x^2} i - \frac{2y}{x} j$; in moving an object from $P(1,1)$ to $Q(4,-2)$.

Solution: $W = \int_C F \cdot dr = f(4,2) - f(1,1) = 0$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{2y}{x^2}$, the force field is conservative. You verify that $f(x,y) = -\frac{y^2}{x}$

30. a) Open b) Not connected c) Not simply connected
32. a) Not open b) not connected c) Not simply connected

Section 16.4 Green's Theorem

Green's Theorem gives the relation between a line integral around and simple closed curve C and a double integral over the plane region D bounded by C.

Green's Theorem: let c be a positively oriented, piece-wise smooth, simple closed curve in the plane and let D be the region bounded by C. If p and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Examples

1. Use Green's theorem to evaluate the line integral along the given positively oriented curves:

a) $\int_C x^2 y^2 dx + 4xy^3 dy$ where C is the triangle with vertices (0,0), (1,3), and (0,3).

Solution: In this case our domain D is described as $D = \{(x,y) | 0 \le x \le 1, 3x \le y \le 3\}$

$\int_C x^2 y^2 dx + 4xy^3 dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_{3x}^3 (4y^3 - 2x^2 y) dy dx = 318/5$

b) $\int_C \sin y dx + x\cos y dy$ where C is the ellipse $x^2 + xy + y^2 = 1$

Solution: In this case our domain D is the region enclosed by the ellipse.

$$\int_C \sin y dx + x\cos y dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0$$

2. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's theorem to prove that the coordinates of the centroid $(\overline{x}, \overline{y})$ of D are given as $\overline{x} = \frac{1}{2A}\oint_C x^2 dy,\ \overline{y} = -\frac{1}{2A}\oint_C y^2 dx$, where A is the area of D.

Solution: By Green's theorem $\frac{1}{2A}\oint_C x^2 dy = \frac{1}{2A}\iint_D 2x dA = \frac{1}{A}\iint_D x dA = \overline{x}$ and

$$-\frac{1}{2A}\oint_C y^2 dy = -\frac{1}{2A}\iint_D (-2y) dA = \frac{1}{A}\iint_D y dA = \overline{y}$$

3. Find the centroid of a semicircular region of radius a

Solution: For the semicircle $A = \frac{1}{2}\pi a^2$. Now

$$\overline{x} = \frac{1}{\pi a^2}\oint_C x^2 dy = \frac{1}{\pi a^2}\left(\int_{-a}^{a} x^2 dy + \int_0^{\pi} a^2 \cos^2 t \cdot a\cos t dt \right) = 0$$

$$\overline{y} = -\frac{1}{\pi a^2}\oint_C y^2 dx = \frac{1}{\pi a^2}\left(\int_{-a}^{a} 0 dx + \int_0^{\pi} a^2 \sin^2 t(-a\sin t dt \right) = \frac{4a}{3\pi}$$

4. Evaluate the line integral $\oint_C xy^2 dx + x^3 dy$, where C is the rectangle with vertices (0,0), (2,0), and (0,3) by a) directly, b) Green's theorem

Solution: a) Directly

$$\oint_C xy^2 dx + x^3 dy = \oint_{C_1+C_2+C_3+C_4} xy^2 dx + x^3 dy$$

$$= 0 + \int_0^3 8dt - \int_0^2 9(2-t)dt + 0$$

where

$$C_1 : x = t,\ y = 0, 0 \le t \le 2$$
$$C_2 : x = 2,\ y = t, 0 \le t \le 3$$
$$C_3 : x = 2 - t,\ y = 3, 0 \le t \le 2$$
$$C_2 : x = 0,\ y = 3 - t, 0 \le t \le 3$$

b) By Green's theorem $\oint_C xy^2 dx + x^3 dy = \int_0^2\int_0^3 (3x^2 - 2xy) dy dx = 6$

5. Evaluate the line integral $\oint_C ydx - xdy$, where C is the circle $x^2 + y^2 = 1$ by a) directly, b) Green's theorem

Solution: a) Directly: use $x = \cos t,\ y = \sin t$ then

$$\oint_C ydx - xdy = \int_0^{2\pi} -\sin^2 tdt - \cos^2 tdt = -2\pi$$

b) Green's theorem: $\oint_C ydx - xdy = \iint_D -2dA = -2\pi(1)^2$

6. Evaluate the line integral $\oint_C xdx + ydy$, where C consists of line segments from (0,1) to (0,0) and from (0,0) to (1,0)and the parabola $y = 1 - x^2$ from (1,0) to (0,1) by a) directly, b) Green's theorem

a) $\oint_C xdx + ydy = \int_0^1 (t-1)dt + \int_0^1 tdt + \int_0^1 (t-1)dt + (2t - t^2)(2-2t)dt = 0$

b) Green's theorem: $\oint_C xdx + ydy = \iint_D 0dA = 0$

7. Use Green's theorem to evaluate $\oint_C F \cdot dr$. Check the orientation before applying Green's theorem. Given that $F =< y^2 \cos x, x^2 + 2y \sin x >$ and C is the triangle from (0,0) to (2,6) to (2,0) to (0,0). Answer: -16

Solution: Hint: Use $\oint_C F \cdot dr = -\int_{-C} y^2 \cos xdx + (x^2 + 2y\sin x)dy$ and apply Green's theorem.

8. A particle starts at the point (-2,0), moves along the x-axis to (2,0), and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by the force field $F =< x, x^3 + 3xy^2 >$

Solution:

$$W = \oint_C F \cdot dr = \int_C xdx + (x^3 + 3xy^2)dy$$

$$= \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} (3x^2 + 3y^2 - 0)dydx$$

$$= 3\int_0^{\pi}\int_0^{2} r^3 drd\theta = 12\pi$$

Section 16.5 Carl and Divergence

Definition: Curl of a vector field F is defined as

$$Curl\ F = \nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k$$ is a vector field.

Where $F(x, y, z) = Pi + Qj + Rk$ a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exists. The short cut form of Curl F can also be written as follows

$$Curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem: If *f* is a scalar function of three variables that has the continuous second order partial derivatives then $curl(grad\ f) = 0 \Rightarrow \nabla \times (\nabla f) = 0$

Note: If F is a conservative vector field then $curl\ F = 0 \Rightarrow \nabla \times F = 0$. If $\nabla \times F \neq 0$ then, F is not conservative.

Definition: Divergence of a vector field F is defined as $div\ F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ which is a scalar quantity. Where $F(x, y, z) = Pi + Qj + Rk$ a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exists.

Theorem: If $F(x, y, z) = Pi + Qj + Rk$ is a vector field on $\mathbb{R}^3$ and the second order continuous partial derivatives of P, Q, and R exists, then $div(curl\ F) = 0 \Rightarrow \nabla \cdot (\nabla \times F) = 0$

Vector form of Green's Theorem: For the vector field $F = P(x, y)i + Q(x, y)j$ verify that $curl\ F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k \Rightarrow curl\ F \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and the Green's theorem becomes of the form $\oint_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (curl\ F) \cdot k\, dA$

Examples

1. Show that the vector field $F(x, y, z) = xzi + xyzj - y^2k$ is not a conservative vector field.

Solution: Verify that $\nabla \times F \neq 0$

2. Given that $F(x,y,z) = e^x \sin y\, i + e^x \cos y\, j + z\, k$. Find curl F and div F.

Solution: Use the definition of curl and div to get $Curl\, F = O$, a zero vector and $div\, F = 1$ a scalar.

3. Is $F(x,y,z) = zy\, i + xz\, j + xy\, k$ a conservative field? If it is find the scalar function *f*.

Solution: Verify that $Curl\, F = O$ and so F is conservative. And we have now that $\nabla f = F = \langle yz, xz, xy \rangle$

Use $f_x = yz \Rightarrow f(x,y,z) = xyz + g(y,z)$ by integrating both sides with respect to x

And again by differentiation with respect to y we have

$f_y = xz + g'(y,z) \Rightarrow g'(y,z) = 0 \Rightarrow g(y,z) = h(z)$. Thus we have

$f(x,y,z) = xyz + h(z) \Rightarrow f_z(x,y,z) = xy + h'(z) \Rightarrow h(z) = K$ a constant. Finally we have $f(x,y,z) = xyz + K$

4. Is there a vector field G on $\mathbb{R}^3$ such that $curl\, G = xy^2\, i + yz^2\, j + zx^2\, k$? Explain.

Solution: use the theorem that for $F(x,y,z) = P\, i + Q\, j + R\, k$ a vector field on $\mathbb{R}^3$ we have $div(curl\, F) = 0 \Rightarrow \nabla \cdot (\nabla \times F) = 0$. So we find that $div(curl\, G) = y^2 + z^2 + x^2 \neq 0$. Thus there is no such function G for which $curl\, G = xy^2\, i + yz^2\, j + zx^2\, k$.

5. For a given vector field F if $\nabla \cdot F = 0$, the field F is incompressible. Show that the field $F = f(y,z)\, i + g(x,z)\, j + h(x,y)\, k$ is incompressible.

Solution: Just show that $div\, F = 0$

6. Let $\overline{r} = x\, i + y\, j + z\, k$, $r = |\overline{r}|$ the magnitude of $\overline{r} = x\, i + y\, j + z\, k$, show that $\nabla \cdot (r\overline{r}) = 4r$

Solution: $\nabla \cdot (r\overline{r}) = (i\,\partial/\partial x + j\,\partial/\partial y\partial + k\,\partial/\partial z) \cdot (rxi + ryj + rzk)$ and now do the rest.

Section 16.6 Parametric surfaces and their areas

No question will be set on test 4 from this section

Section 16.7 Surface Integrals

Formulas for this section:

1. $\iint_S f(x,y,z)\,dS = \iint_S f(x,y,g(x,y))\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,dA$
2. $\iint_S f(x,y,z)\,dS = \iint_S f(r(u,v))\,|r_u \times r_v|\,dA$
3. $\iint_S F(x,y,z) \cdot dS = \iint_S F(x,y,z) \cdot n\, dS$

Examples:

1. Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2, 0 \le x \le 1,\ 0 \le y \le 2$

Solution: We have $\partial z / \partial x = 1, \partial z / \partial y = 2y$ and then

$$\iint_S ydS = \iint_D y\sqrt{1+1+(2y)^2}\,dA = \int_0^1\int_0^2 y\sqrt{2+4y^2}\,dydx = 13\sqrt{2}/3$$

2. Evaluate $\iint_S x^2\,dS$, where S is the unit sphere $z^2 + x^2 + y^2 = 1$

Solution: Use spherical coordinate system
$x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\phi, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$,
$r(\phi,\theta) = xi + yj + zk = i\sin\phi\cos\theta + j\sin\phi\sin\theta + k\cos\phi,$ now find $\left|r_\phi \times r_\theta\right| = \sin\phi$

$$\iint_S x^2\,dS = \iint_D (\sin\phi\cos\theta)^2 \sin\phi dA = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta d\phi d\theta = 4\pi/3$$

3. Find the flux of the vector field $F(x,y,z) = z\,i + y\,j + x\,k$ across the unit sphere $z^2 + x^2 + y^2 = 1$.

Solution: Use $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\phi, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ and
$r(\phi,\theta) = xi + yj + zk = i\sin\phi\cos\theta + j\sin\phi\sin\theta + k\cos\phi,$
$r_\phi \times r_\theta = i\sin^2\phi\cos\theta + j\sin^2\phi\sin\theta + k\sin\phi\cos\phi$
$F(r(\phi,\theta) = i\cos\phi + j\sin\phi\sin\theta + k\sin\phi\cos\theta$
Now

$$\iint_S F(x,y,z)\cdot dS = \iint_D F\cdot n\,dS$$

$$= \iint_D F\cdot(r_\phi \times r_\theta)dA = \int_0^{2\pi}\int_0^{\pi}(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)d\phi d\theta$$

$$= 4\pi/3$$

4. Evaluate $\iint_S y\,dS$ where S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}),\ 0 \le x \le 1, 0 \le y \le 1$

Solution: $\iint_S y\,dS = \iint_D y\sqrt{1+y+x}dA = \int_0^1\int_0^1 y\sqrt{1+x+y}dxdy = 4/105(9\sqrt{3} + 4\sqrt{2} - 2)$

5. Evaluate $\iint_S F\cdot dS,\ F = xi + yj + z^4k$, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

Solution: Use

$$\iint_S F\cdot dS = -\iint_D (-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R)dA$$
$$= \iint_D (\sqrt{x^2+y^2} - (x^2+y^2)^2)dA$$
$$= \int_0^{2\pi}\int_0^1 (r-r^4)rdrd\theta = \pi/3$$

Section 16.8 Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let also that F be a vector field whose components have continuous partial derivatives on an open region in 3-D that contains S. Then

$$\iint_S F\cdot dr = \iint_S curl\, F\cdot dS = \iint_D curl\, F\cdot kdA$$

Examples

1. Evaluate $\iint_S F\cdot dr,\ F = -y^2 i + xj + z^2 k$, where C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$

Solution: Find that $curl\, F = (1+2y)k$

$$\iint_S F\cdot dr = \iint_S curl\, F\cdot dS = \iint_D curl\, F\cdot kdA = \iint_D (1+2y)dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)rdrd\theta = \pi$$

2. $F(x,y,z) = \langle ax^3 - 3xz^2, x^2y + by^3, cz^3 \rangle$, where C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2,$ and the cylinder $x^2+y^2=1$ oriented counter clockwise as viewed from above, and C is the boundary of S. find a, b, c for which $\iint_S F\cdot dS$ is independent of the choice of S.

Solution: Suppose G be any vector field such that $F = curl\, G$, then $\iint_S F\cdot dS = \iint_D curl\, G\cdot dS$ depends only on the values of S and div curl G = 0.

So we have

$$divF = 0 \Rightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$
$$\Rightarrow (3a+1)x^2 + 3by^2 + (3c-3)z^2 = 0 \Rightarrow a = -1/3, b = 0, c = 1$$

Section 16.9 Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (upward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S F \cdot dS = \iiint_E div\, F\, dV$$

Example 1. Find the flux of the vector field $F(x, y, z) = zi + yj + xk$ over the unit sphere $x^2 + y^2 + z^2 = 1$

Solution: $\iint_S F \cdot dS = \iiint_E div\, F\, dV = \iiint_E 1 dV = 4\pi / 3$