DEPARTMENT OF MATHEMATICS AND STATISTICS

COMPLETE

Chapter 15 Multiple Integrals

Section 15.1 Double Integrals Over Rectangles

Examples:

1. Evaluate the iterated integral a) $\iint_R (5-x)dA,\ R=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$ and b) $\iint_R (4-2y)dA,\ R=[0,1]\times[0,1]$

Solution: a) $\iint_R (5-x)dA = \int_0^1\int_0^1 (5-x)dydx = [5x - x^2/2]_0^1(1) = 4.5$

b) $\iint_R (4-2y)dA = \int_0^1\int_0^1 (4-2y)dydx = [4y - y^2]_0^1 = 3$

2. If $R=[0,1]\times[0,1]$, show that $0 \le \iint_R \sin(x+y)dA \le 1$

Solution: On R, $0\le x+y\le 2<\pi, \sin(x+y)\ge 0$. Thus we have
$0 \le \iint_R \sin(x+y)dA \le \iint_R dA = 1$

Section 15.2 Iterated Integrals

Fubini's Theorem If $f(x,y)$ is differentiable on $R=\{(x,y)\,|\,a\le x\le b, c\le y\le d\}$ then $\iint_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$

Examples

Evaluate the integral in two different ways using Fubini's theorem (1-4).

1. $\int_1^4\int_{-1}^1 (x^2+y^2)dydx = \int_1^4 [x^2y + y^3/3]_{y=-1}^{y=1}dx = [2x^3/3 + 2/3x]_1^4 = 44$

also $\int_{-1}^{1}\int_{1}^{4}(x^2+y^2)dxdy = \int_{-1}^{1}[x^3/3+xy^2]_{x=1}^{x=4}dy = [21y+y^3]_{-1}^{1} = 44$

2. $\int_0^1\int_1^2 \frac{xe^x}{y}dydx$

3. $\int_1^2\int_0^1 (x+y)^{-2}dxdy$

4. $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}dydx$

5. Evaluate $\iint_R \cos(x+2y)dA,\ R=\{(x,y)\,|\,0\le x\le \pi, 0\le y\le \pi/2\}$

6. Evaluate $\iint_R \frac{1+x^2}{1+y^2}dA,\ R=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$

7. Find the volume of the solid in the first octant bounded by the cylinder $z=9-y^2$ and the plane $x=2$

Solution: On the xy plane $z=0 \Rightarrow y^2=9 \Rightarrow y=3$ in the first octant. Thus we have

$$V=\int_0^2\int_0^3 (9-y^2)dydx = 36$$

8. Find the average value of $f(x,y)=e^y\sqrt{x+e^y}, R=[0,4]\times[0,1]$.

   The average value is given by $f_{ave}=\frac{1}{A(R)}\iint_R f(x,y)dA$, where A(R) represents the area of the region R.

   Now $f_{ave}=\int_0^4\int_0^1 e^y\sqrt{x+e^y}dydx=$ (Try to solve the integral)

9. Evaluate $\int_0^3\int_0^5 (5-x)dxdy$. Ans: 37.5

10. From problem 9. draw the bounded region by $z=5-x, R=\{(x,y)\,|\,0\le x\le 5, 0\le y\le 3\}$ and evaluate $\int_0^3\int_0^5 (5-x)dxdy$ geometrically.

11. Evaluate $\int_1^3\int_2^4 (40-2xy)dydx$. Ans: 112

12. Evaluate $\iint_R y^2x dA$ over the rectangle $R=\{(x,y)\,|\,-3\le x\le 2, 0\le y\le 1\}$.

    Ans: -5/6

13. Evaluate $\iint_R x\cos(xy)\cos^2(\pi x)dA;\ R=[0,1/2]\times[0,\pi]$ Ans: $1/(3\pi)$.

14. Evaluate

Section 15.3 Double Integral Over General Regions

Examples

1. Evaluate $\int_0^1 \int_{-x}^{x^2} xy^2\,dydx$ Answer: 13/120

2. $\int_0^1 \int_0^v \sqrt{1-v^2}\,dudv$ Answer: 1/3

3. $\iint_R xydA$ over the region R enclosed between $y = 1/2x, y = \sqrt{x}, x = 2, x = 4$

   Answer: 11/6

4. $\iint_D (2x - y^2)dA,\ D = \{(x, y) \mid y = -x + 1, y = x + 1, y = 3\}$ Answer: -68/3

5. Use double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane given by $z = 4 - 4x - 2y$

Hint: $V = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)dydx$ Answer: 4/3

6. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4, z = 0$ Answer: 16π

7. Evaluate the following multiple integrals:

   a) $V = \int_0^1 \int_{x^3}^{x^2} (2x + y^2)dydx$ Answer: 4/35

   b) $V = \int_0^1 \int_{\sqrt{y}}^{y^{1/3}} (2x + y^2)dxdy$ Answer: 4/35

8. Evaluate $\iint_R (2x + y^2)dA$, where R is bounded by the curves $y = x^2, y = x^3$ and compare your result with example 7. Answer: 4/35

9. Sketch the region of integration and change the order of integration of

   $\int_0^4 \int_0^{x^{1/2}} f(x, y)dydx$ Answer: $\int_0^2 \int_{y^2}^4 f(x, y)dxdy$

10. Homework # 20: $V = \int_0^1 \int_{y^3}^{y^2} (2x + y^2)dxdy = 19/210$

11. Homework # 32: $V = \int_{-1}^1 \int_{x^2}^1 (2 + y)dydx - \int_{-1}^1 \int_{x^2}^1 3ydydx = 16/15$

12. Homework # 38: $V = \int_0^1 \int_{4x}^4 f(x, y)dydx = \int_0^4 \int_0^{y/4} f(x, y)dxdy$

13. Homework # 56: $\iint_D (2-3x+4y)dA = \iint_D 2dA = 2A(D) = 2(50) = 100$ using symmetry.

Section 15.4 Polar Coordinates

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b,\ \alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$ then $\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r,\theta) r dr d\theta$

Examples

1. Evaluate $\iint_R \sin\theta\, dA$, where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$

Hint: $\iint_R \sin\theta\, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta r dr d\theta = 8/3$

2. The sphere of radius a centered at origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$ and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

Hint: $r^2 + z^2 = a^2 \Rightarrow z = \pm\sqrt{a^2 - r^2}$, $V = 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\, r dr d\theta = 4/3\pi a^3$

3. Use polar coordinate to evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$

Hint: draw a diagram and set $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx = \int_0^{\pi}\int_0^1 r^4 dr d\theta = \pi/5$

4. Homework Problems:

4) $\iint_R f(x,y)\, dA = \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta) r dr d\theta$

17) One loop of the rose $r = \cos 3\theta$: $A = \iint_D dA = \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r dr d\theta = \pi/12$

18) Area of the region enclosed by $r = 4 + 3\cos\theta$,

$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 4 + 3\cos\theta\}$, $A = \iint_D dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r dr d\theta = 41/2\pi$

Section 15.7 Triple Integrals

Examples

1. Evaluate $\iiint_E 12xy^2z^2\,dv$ where E is defined as a region bounded by $-1 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 2$.

Hint: Draw he bounded region and set $\iiint_E 12xy^2z^2\,dv = 12\int_{-1}^{2} x\,dx \int_0^3 y^2\,dy \int_0^2 z^2\,dz = 648$

2. Evaluate $\iiint_E z\,dv$ where E is the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x,\ x = 0$

Hint: Draw diagram and set $\iiint_E z\,dv = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = 1/8$ using type I region

3. Evaluate $\iiint_E z\,dv$ where E is the solid tetrahedron bounded by the four planes $y = 0,\ x = 0, z = 0,\ \ x + y + z = 1$

Hint: $V = \iiint_E z\,dv = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx = 1/24$

4. Evaluate $\iiint_E y\,dv$ where E is the solid tetrahedron bounded by the four planes $y = 0,\ x = 0, z = 0,\ \ 2x + 2y + z = 4$ Answer: V = 4/3
5. Use triple integral to find volume of the tetrahedron bounded by the four planes $2y = x,\ x = 0, z = 0,\ \ x + 2y + z = 2$ Answer: V = 1/3
6. Find the volume of the tetrahedron with vertices (0, 0, 0), (1, 2, 0), (0, 2, 2) and (0, 2, 0). Use Type I, Type II and Type III regions.

Hint: Find the plane through (0, 0, 0), (1, 2, 0), and (0, 2, 2) which is $z = y - 2x$.

Type I: $V = \int_0^1\int_{2x}^2\int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx = \int_0^2\int_0^{y/2}\int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy$, from xy – solid

Type II: $V = \int_0^2\int_0^2\int_0^{1/2(y-z)} f(x,y,z)\,dx\,dy\,dz = \int_0^2\int_0^y\int_0^{1/2(y-z)} f(x,y,z)\,dx\,dz\,dy$, from zy - solid

Type III: $V = \int_0^2\int_0^{-1/2z}\int_{z+2x}^2 f(x,y,z)\,dy\,dx\,dz = \int_0^1\int_0^{-2x}\int_{z+2x}^2 f(x,y,z)\,dy\,dz\,dx$, from zx - solid

7. Homework Problems:

12) $\iiint_E zx\,dv = \int_0^1\int_0^y\int_0^{y-z} zx\,dx\,dz\,dy = 1/120$, use the plane $z = y - x$ thru (0, 0, 0), (1, 1, 0) and (0, 1, 1)

16. $\iiint_E z\,dv = \int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = 27/8$

30. $\iiint_E f(x,y,z)\,dv = \int_{-1/3}^{1/3}\int_{-1/2\sqrt{1-9x^2}}^{1/2\sqrt{1-9x^2}}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f\,dz\,dy\,dx$

$$= \int_{-1/2}^{1/2}\int_{-1/3\sqrt{1-4y^2}}^{1/3\sqrt{1-4y^2}}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f\,dz\,dx\,dy$$

$$= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}\int_{-1/3\sqrt{1-z^2-4y^2}}^{1/3\sqrt{1-z^2-4y^2}} f\,dx\,dz\,dy$$

$$= \int_{-1}^{1}\int_{-1/2\sqrt{1-z^2}}^{1/2\sqrt{1-z^2}}\int_{-1/3\sqrt{1-z^2-4y^2}}^{1/3\sqrt{1-z^2-4y^2}} f\,dx\,dy\,dz$$

$$= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}}\int_{-1/2\sqrt{1-z^2-9x^2}}^{1/2\sqrt{1-z^2-9x^2}} f\,dy\,dz\,dx$$

$$= \int_{-1}^{1}\int_{-1/3\sqrt{1-z^2}}^{1/3\sqrt{1-z^2}}\int_{-1/2\sqrt{1-z^2-9x^2}}^{1/2\sqrt{1-z^2-9x^2}} f\,dy\,dx\,dz$$

34. Type I, xy solid

$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x^2, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1, 0 \le z \le y\}$

Type II, zy solid

$E = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y, \sqrt{y} \le x \le 1\} = \{(x,y,z) \mid 0 \le z \le 1,\ z \le y \le 1, \sqrt{y} \le x \le 1\}$

Type III, xz solid

$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le z \le x^2, z \le y \le x^2\} = \{(x,y,z) \mid 0 \le z \le 1,\ \sqrt{z} \le x \le 1, z \le y \le x^2\}$

48. $f_{ave} = \dfrac{1}{V(E)}\iiint_E f(x,y,z)dv$, where $V(E) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} dz\,dy\,dx = \pi/2$ using polar coordinate. Now

$$f_{ave} = \frac{1}{V(E)}\iiint_E f(x,y,z)dv = \frac{2}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} 2(x^2+y^2)\,dz\,dy\,dx = 1/12$$

Section 15.8 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates:

$$\iiint_E f(x,y,z)dv = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)rdzdrd\theta$$

Spherical Coordinates:

$$\iiint_E f(x,y,z)dv = \int_c^d\int_{\alpha}^{\beta}\int_a^b f(\rho,\theta,\phi)\rho^2\sin\phi d\rho d\theta d\phi$$

Examples

1. Use triple integral in cylindrical coordinates to find the volume of the solid E that is bounded by the hemisphere $z=\sqrt{25-x^2-y^2}$ below bounded by the xy – plane and laterally by the cylinder $x^2+y^2=9$

Hint: $\iiint_E z\,dv = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} dzdydx = \int_0^{2\pi}\int_0^3\int_0^{\sqrt{25-r^2}} rdzdrd\theta = 122\pi/3$

2. Evaluate by cylindrical coordinate: $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2dzdydx$

Hint: $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2dzdydx = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta dzdrd\theta = 243\pi/4$

3. Use spherical coordinate to evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}dzdydx$

Hint: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}dzdydx = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^5\cos^2\phi\sin\phi d\rho d\phi d\theta = 64\pi/9$

4. Evaluate $\iiint_E (x^3+xy^2)dv$ where E is the solid in the first octant that lies beneath the paraboloid $z=1-x^2-y^2$

Hint: $E=\{(r,\theta,z)\,|\,0\le\theta\le\pi/2, 0\le r\le 1, 0\le z\le 1-r^2\}$ and

$$\iiint_E (x^3+xy^2)dv = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta dzdrd\theta = 2/35$$

5. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$, and the sphere $x^2+y^2+z^2=4$

Hint: $E=\{(r,\theta,z)\,|\,0\le\theta\le 2\pi, 0\le r\le 1, -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\}$ and

$$V=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} rdzdrd\theta = 4\pi/3(8-3\sqrt{3})$$

6. Use spherical coordinates to evaluate $\iiint_H (x^2+y^2)dv$ where H is the hemispherical region that lies above the xy plane and below the sphere $x^2+y^2+z^2=1$

Hint: $\iiint_H (x^2+y^2)dv = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4 \sin^3\phi d\rho d\theta d\phi = 4\pi/15$

7. Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}dxdydz$

Hint: Use spherical coordinate and find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}dxdydz = \lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho^3 e^{-\rho^2}\sin\phi d\rho d\theta d\phi = 2\pi$$

8.

Section 15.9 Change of Variables in Multiple Integrals

Examples

1. Given $x = 1/4(u+v),\ y = 1/2(u-v)$ and T is the transformation from uv plane to xy plane. Find

   a) $T(1,3)$

   b) Sketch the constant v – curve corresponding to $v = -2, -1, 0, 1, 2$

   c) Sketch the constant u – curve corresponding to $u = -2, -1, 0, 1, 2$

   d) Sketch the image of the square region in uv plane under the transformation T to the xy plane.

   Hint: a) $T(u,v) = (x,y) \Rightarrow u = 1, v = 3 \Rightarrow x = 1, y = -1$, thus $T(1,3) = (1,-1)$
   b) Solving for u and v we have $u = 2x+y,\ v = 2x-y$
   For given $v = -2, -1, 0, 1, 2$ we find
   $2x-y=-2, 2x-y=-1, 2x-y=0, 2x-y=1, 2x-y=2$. You can plot all these equations in the xy plane.
   c) We have $u = 2x+y$. For given $u = -2, -1, 0, 1, 2$ we find
   $2x+y=-2, 2x+y=-1, 2x+y=0, 2x+y=1, 2x+y=2$. You can plot all these equations in the xy plane.

   d) Try yourself.

2. Evaluate $\iint_R \frac{x-y}{x+y}dA$, where R is the region bounded by

   $x-y=0, x-y=1, x+y=1, x+y=3$.

Hint: Direct evaluation is complicated, we use substitution like
$u=x+y,\ v=x-y \Rightarrow u=1,3,\ v=0,1\ and\ x=1/2(u+v), y=1/2(u-v)$

The Jacobian $J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$

Now $\iint_R \frac{x-y}{x+y}dA = \int_0^1\int_1^3 \frac{u}{v}\left|J(u,v)\right| dudv = 1/4\ln 3$

3. Evaluate $\iint_R e^{xy}dA$, where R is the region bounded by

   $y=1/2x,\ y=x,\ y=1/x,\ y=2/x$

Hint: Write $u=y/x,\ v=xy$, $\iint_R e^{xy}dA = \int_{1/2}^{1}\int_1^2 e^v \frac{1}{2u} dvdu = 1/2(e^2-e)\ln 2$

4. Find the volume of the region enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Hint: Use the transformation (substitution)
$u=x/a, v=y/b,\ w=z/c \Rightarrow u^2+v^2+w^2=1$, which is a sphere of radius 1. Now
$\iiint_R dV = \iiint_S \left|J(u,v,w)\right| dudvdw = 4\pi abc/3$. For volume of a sphere see example 2, section 15.4.

5. Find the image of the set S which is a disk given by $u^2+v^2\le 1$ under the transformation $x=au, y=bv$.

6. Evaluate $\iint_R (x-3y)dA$, where R is the triangular region with vertices (0, 0), (2, 1) and (1, 2).

   Hint: The line thru (0, 0) and (2, 1) is $y=1/2x$, which is the image of v =0
   The line thru (0, 0) and (1, 2) is $y=2x$, which is the image of u =0 and the line thru (1, 2) and (2, 1) is $x+y=3$, which is the image of $u+v=1$

   The Jacobian $J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

   Now $\iint_R (x-3y)dA = \int_0^1\int_0^{1-u} (-u-5v)3dvdu = -3$

7. Evaluate $\iint_R (x^2 - xy + y^2)dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$; under the substitution $x = \sqrt{2}u - \sqrt{2/3}v,\ y = \sqrt{2}u + \sqrt{2/3}v$

Solution: $x^2 - xy + y^2 = 2 \Rightarrow u^2 + v^2 = 1$, The Jacobian $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 4/\sqrt{3}$

$$\iint_R (x^2 - xy + y^2)dA = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 2(u^2 + v^2)4/\sqrt{3}\, dudv = 4\pi/\sqrt{3}$$

8. Evaluate $\iint_R \cos\frac{y-x}{y+x}dA$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, 2), (0, 1).

   Hint: Use $u = y - x, v = y + x,\ J(u,v) = -1/2$, $\iint_R \cos\frac{y-x}{y+x}dA = 3/2\sin 1$

9. Evaluate $\iint_R \sin(9x^2 + 4y^2)dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 1$. Use $u = 3x, v = 2y,\ J(u,v) = 1/6$

   $$\iint_R \sin(9x^2 + 4y^2)dA = 1/6\iint_S \sin(u^2 + v^2)dudv = 1/6\int_0^{\pi/2}\int_0^1 r\sin r^2 dr d\theta$$