ARIZONA STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Chapter 14 Partial Derivatives

Section 14.1 Functions of Several Variables

Definition: A function *f* of two variables is a rule that assigns to each ordered pair of real numbers (*x*, *y*) in a set D a unique real number denoted by *f*(*x, y*). The set D is the domain of *f* and its range is the set of values that *f* takes on, that is $\{f(x, y) \mid (x, y) \in D\}$.

Definition: If *f* is a function of two variables with domain D, then the graph of f is the set of all points (*x*, *y*, *z*) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

Definition: The **Level Curves (contour curve)** of a function *f* of two variables are the curves with equation $f(x, y) = k$ where k is a constant (in the range of *f*)

Functions of three or more variables: A function of three variables *f*, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$

Examples:

1. Find the domain of a) $f(x, y) = \dfrac{x + y + 1}{x - y}$ and b) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - y}$

   Solution: a) The domain of *f* is $D = \{(x, y) \mid x \neq y\}$
   b) The domain of *f* is $D = \{(x, y) \mid x + y + 1 \geq 0, x \neq y\}$

2. Find the domain and range of $f(x, y) = \sqrt{36 - x^2 - y^2}$

   Solution: The domain of *f* is $D = \{(x, y) \mid x^2 + y^2 \leq 36\}$ that is all points inside and on the circle of radius 6.
   And the range of the function of *f* is $\{z \mid z = \sqrt{36 - x^2 - y^2}, (x, y) \in D\}$

3. Find the domain and range of *f* is $z = h(x, y) = 4x^2 + y^2$

   Solution: We have seen in chapter 13 that the function h(x, y) is an elliptic paraboloid with vertex at (0, 0, 0), and opens upward. Horizontal traces are ellipses and vertical traces are parabolas. The domain is all the ordered pairs (*x, y*) in $\mathbb{R}^2$, that is the xy-plane. The range is the set $[0,\infty)$ of all nonnegative real numbers.

4. Sketch all the level curves of the function $f(x,y)=\sqrt{36-x^2-y^2}$ for $k=0,1,2,3$
5. Find the level surfaces of the function $f(x,y,z)=x^2+y^2+z^2$

Solution: Choose different numerical values of $f(x,y,z)$ and observe that $k=x^2+y^2+z^2$ represents spheres as level surfaces. See example 15, page # 897 at your text.

Section 14.2 Limits and Continuity

Definition: Let $f(x,y)$ be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then the limit of $f(x,y)$ as (x,y) approaches (a,b) is *L* and is written as s and continuity

$$\lim_{(x,y)\to(a,b)} f(x,y)=L$$

if for every number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that
$|f(x,y)-L|<\varepsilon$ whenever $(x,y)\in D$ and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$

Definition: Continuous function The function $f(x,y)$ is continuous on D if f is continuous at every point (a,b) in D.

Examples:

1. Given $f(x,y)=\dfrac{xy}{x^2+y^2}$. Find the limits when $(x,y)\to(0,0)$ along

a) the *x* axis b) the *y* axis c) the line *y* = *x* d) the line *y* = -*x*
e) the parabola $y=x^2$

Solution: a) Along x axis y = 0: $\lim_{(x,y)\to(0,0)} f(x,0)=\dfrac{x(0)}{x^2+0^2}=0$

b) Along y axis x = 0: $\lim_{(x,y)\to(0,0)} f(0,y)=\dfrac{(0)y}{0^2+y^2}=0$

c) Along the line *y = x:* $\lim_{(x,y)\to(0,0)} f(x,x)=\dfrac{x^2}{x^2+x^2}=1/2$

d) Along the line *y =- x:* $\lim_{(x,y)\to(0,0)} f(x,-x)=\dfrac{x^2}{x^2+x^2}=1/2$

e) Along the parabola $y = x^2$ $\lim\limits_{(x,y)\to(0,0)} f(x,x^2) = \dfrac{x^3}{x^2+x^4} = 0$

2. Given $f(x,y) = \dfrac{xy}{x^2+y^2}$. Find the limit if exists.

In Example 1, we have seen different values along different lines/curves, thus the limit does not exist.

3. Evaluate $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy}{x^2+y^2}$ Answer: DNE

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

Solution: $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim\limits_{r\to0^+} r^2 \ln r^2 = \lim\limits_{r\to0^+} \dfrac{2\ln r}{1/r^2} = \lim\limits_{r\to0^+} \dfrac{2}{-1/r^2} = 0$

Where $x^2+y^2 = r^2$

5. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

Solution: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 0}{2x^2+0^2} = 1/2$ along x axis and $\lim\limits_{(x,y)\to(0,0)} \dfrac{0^2+\sin^2 y}{2(0)^2+y^2} = 1$. Limit DNE.

6. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$ Answer: DNE

7. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$ Answer: 0

8. Homework problems:

10. $\lim\limits_{(x,y)\to(0,0)} \dfrac{6x^3y}{2x^4+y^4}$ Answer: DNE

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^4}{x^2+y^8}$ Answer: DNE

18. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$ Answer: DNE

22. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$ Answer: DNE

36. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2}, & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$

The limit $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{x^2+xy+y^2}$ does not exist, therefore it is discontinuous.

Section 14.3 Partial Derivatives

Examples;

1. Find $f_x(1,3),\ f_y(1,3), f_{xy}(1,3), f_{yx}(1,3), f_{xx}(1,3), f_{yy}(1,3)$ for
   $f(x,y)=2x^2y+2y+4x$

Solution: $f_x(1,3)=\dfrac{\partial f(1,3)}{\partial x}=54x^2+4=58$, $f_y(1,3)=\dfrac{\partial f(1,3)}{\partial y}=4y+2=14$. You can find the rest

2. $f(x,y)=x^5+3x^3y^2+3xy^4$, find $f_x,\ f_y$
3. $f(x,y)=xe^{-x^2-y^2}$, find $f_x,\ f_y$
4. $f(x,t)=\arctan(x\sqrt{t})$, find $f_x,\ f_t$
5. $f(x,y)=\ln(3x+5y)$, find $f_x,\ f_y, f_{xx}, f_{yy}, f_{xy}$

Section 14.4 Tangent Planes and Linear Approximations

Suppose the function $f(x,y)$ has continuous partial derivatives. An equation of the tangent plane to the surface $z=f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Examples:

1. Given $f(x,y)=\dfrac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$, find equation of the tangent plane at (1,1,1)

Solution: The equation is $z-1=f_x(1,1)(x-1)+f_y(1,1)(y-1)$

Where $f_x(1,1)=f_y(1,1)=-8/9$

2. $w=xye^{xz}$, find $dw=w_xdx+w_ydy+w_zdz$

Section 14.5 The Chain Rule

Forms:

1. Given $y = f(x), x = g(t),$ then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$
2. Given $z = f(x, y), x = g(t), y = h(t)$ then $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
3. Given $z = f(x, y), x = g(s,t), y = h(s,t)$ then $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$
4. Implicit differentiation: $\frac{dy}{dx} = -\frac{F_x}{F_y}, F(x, y) = 0, y = f(x)$

Examples:

1. Given $z = xy^2 + 4x^3y,\ x = \cos 2t, y = \sin t,$ find $\frac{dz}{dt}$
2. Given $z = \tan^{-1}(2x + y),\ x = s^2t, y = s\ln t,$ find $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$

Solution: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{4st}{1+(2x+y)^2} + \frac{\ln t}{1+(2x+y)^2}$ and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2s^2}{1+(2x+y)^2} + \frac{s/t}{1+(2x+y)^2}$$

3. Given

$$z = f(x, y), x = g(t), y = h(t), g(3) = 2, h(3) = 7, g'(3) = 5, h'(3) = -4$$

$$f_x(2,7) = 6, f_y(2,7) = -8, \text{find } \frac{dz}{dt},\ t = 3$$

Solution: x and y are functions of one variable only. We have $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

When t = 3, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)h'(3) = 62$

4. Given $M = xe^{y-z^2}$, $x = 2uv, y = u - v, z = u + v$ find $\frac{\partial M}{\partial u}, \frac{\partial M}{\partial v}$ when $u = 3, v = -1$

Solution: use $\frac{\partial M}{\partial u} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u}$ and $\frac{\partial M}{\partial v} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v}$

Section 14.6 Directional Derivatives

In this section we extend the concept of a partial derivative to the more general notion of a directional derivative. The partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate axes. Directional derivatives allow us to compute the rates of change of a function with respect to distance in any direction.

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in the directions parallel to the x-axis and y-axis. In this section we investigate the rates of change of $f(x, y)$ in other directions given.

Definition: If $f(x, y)$ is a function of x and y and a given unit vector $u = u_1 i + u_2 j$, then the directional derivative of $f(x, y)$ in the direction of u at (x_0, y_0) is given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 = \nabla f \cdot u$$

where $\nabla f =< f_x, f_y >$ called the gradient of $f(x, y)$. The gradient of a scalar function is a vector which is orthogonal to the level curves.

Maximum directional derivative: We have $D_u f(x_0, y_0) = \nabla f \cdot u = |\nabla f||u|\cos\theta$ where θ is the angle between the gradient vector and the given unit vector. For maximum value of directional derivative $\cos\theta$ must be equal to 1, which occurs when $\theta = 0$ and then the unit vector has the same direction as the gradient vector.

Theorem: Suppose $f(x, y)$ is a differentiable function of two variables. The maximum value of the directional derivative $D_u f(x, y)$ is $|\nabla f|$ and it occurs when u has the same direction as the gradient vector $\nabla f =< f_x, f_y >$

Tangent Plane and Normal Vector: Assume that $F(x, y, z)$ continuous first order partial derivatives and let $c = F(x_0, y_0, z_0)$. If $\nabla F(x_0, y_0, z_0) \neq O$ then $\nabla F(x_0, y_0, z_0)$ is a normal vector to the surface $c = F(x, y, z)$ at the point $P_0(x_0, y_0, z_0)$ and the tangent plane to the surface at $P_0(x_0, y_0, z_0)$ is

$$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$$

Examples

1. Find the directional derivatives:

a) $f(x, y) = xy,\ u = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$ at the point (1, 2)

b) $f(x, y) = e^{xy},\ u = \cos\theta i + \sin\theta j,\ \theta = \pi/3$ at the point (-2, 0)

c) $f(x, y, z) = x^2 y - yz^3 + z,\ a = 2i + j - 2k$ at the point (1, -2, 0)

d) $f(x, y, z) = \dfrac{y}{x+z}$, at the point P(2, 1, -1) in the direction from P to Q(-1, 2, 0)

2. Suppose that $D_u f(1,2) = -5,\ D_v(1,2) = 10, u = \frac{3}{5}i - \frac{4}{5}j, v = \frac{4}{5}i + \frac{3}{5}j.$ find $f_x(1,2), f_y(1,2),$ and $D_u f(x, y)$ in the direction of origin.
3. $f(x, y) = x^2 e^y$, find $\max D_u f(-2,0)$
4. $f(x, y) = x^2 e^y$, find $\min D_u f(-2,0)$
5. Find the equation of the tangent plane to $x^2 + 4y^2 + z^2 = 18$ at P(1, 2, 1) and determine the acute angle that the plane makes with the xy plane.

Solution: $\nabla F(x, y, z) = <2x, 8y, 2z>$. Now $\nabla F(1,2,1) = <2,16,2>$
The plane thru P(1, 2, 1) has the equation $2(x-1) + 16(y-2) + 2(z-1) = 0$
The angle between two planes is the angle between the normal to the planes. Let us call the normals $\nabla F(1,2,1) = <2,16,2> = n_1$ and on xy plane $<0,0,1> = n_2$. The angle $\theta = \cos^{-1}\left(\dfrac{n_1 \cdot n_2}{|n_1||n_2|}\right) = \cos^{-1}(1/\sqrt{66})$

Section 14.7 Maximum and Minimum Values

Definition: A function of two variables has local max at (a,b) if $f(x, y) \le f(a,b)$ where (x, y) is near (a,b). The number $f(a,b)$ is called the local maximum value of f
On the other hand if $f(x, y) \ge f(a,b)$ where (x, y) is near (a,b). The number $f(a,b)$ is called the local minimum value of f

Theorem: If f has a local maximum or minimum at (a,b) and the first partial derivatives of f exist then $f_x(a,b) = 0, f_y(a,b) = 0$. The point (a,b) is called a stationary point or a critical point.

Second derivative test: Suppose the second partial derivatives of f are continuous on a disk with center at (a,b), and suppose that $f_x(a,)b = 0, f_y(a,b) = 0$. Let us define that

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

1) If $D > 0,\ and\ f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum
2) If $D > 0,\ and\ f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum
3) If $D < 0, then$, then $f(a,b)$ is neither a local minimum nor a local maximum. In this case the point (a,b) is called a SADDLE POINT. And graph of f crosses its tangent plane at (a,b)

Finding Absolute extrema on a closed and bounded set R

Step 1 Find the critical points of f that lie in the interior of R
Step 2 Find all boundary points at which the absolute extrema can occur
Step 3 Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest of these values is the absolute minimum.

Extreme Value Theorem: If f is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains both an absolute max and an absolute min on D.

Examples:

1. The surface $z = f(x, y) = x^2 + y^2$ has relative min (absolute min) at (0, 0)
2. The surface $z = f(x, y) = 1-(x^2 + y^2)$ has relative max (absolute max) at (0, 0)
3. The surface $z = f(x, y) = \sqrt{x^2 + y^2}$ has relative min (absolute min) at (0, 0)
4. The surface $z = f(x, y) = 3x^2 - 2xy + y^2 - 8y$ has relative min at (2, 6)
5. The surface $z = f(x, y) = x^4 + y^4 - 4xy + 1$ has relative min at (1, 1) and at (-1, -1) and a saddle point at (0, 0)
6. Find all absolute max and min of $z = f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices P(0, 0), Q(3, 0) and M(0, 5)

Section 14.8 The Lagrange Multiplier Method

To find maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, where $\nabla g \neq O$ on the surface $g(x, y, z) = k$

1. Find all values of x, y, z and λ such that
   $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \; and \; g(x, y, z) = k$
2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the absolute maximum and the smallest of these values is the absolute minimum.

Examples:

1. At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?
   Answer: $(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})$, ½
2. Use Lagrange multiplier method to prove that the triangle with maximum area that has a given perimeter P is equilateral.

Solution: $p = x + y + z,\ s = p/2$, where s is the half of perimeter = constant. We need to maximize area $A = \sqrt{s(s-x)(s-y)(s-z)}$. Let us consider
$f(x,y,z) = A^2 = s(s-x)(s-y)(s-z),\ x + y + z = p$, where $g(x,y) = x + y + z$. Now
$\nabla f = \lambda \nabla g \Rightarrow -s(s-y)(s-z) = \lambda,\ -s(s-x)(s-z) = \lambda \Rightarrow x = y$

$$and\ -s(s-x)(s-z) = \lambda, -s(s-x)(s-y) = \lambda \Rightarrow y = z$$

3. Use Lagrange multiplier method to find the point on the plane $x - y + z = 4$ that is closest to the point $(1,2,3)$

Solution: Let us consider a point (x,y,z) on the given plane. To minimize
$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$ with $x - y + z = 4$. We consider
$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2, g(x,y,z) = x - y + z$
Now set
$\nabla f = \lambda \nabla g \Rightarrow 2(x-1) = \lambda, 2(y-2) = \lambda, 2(z-3) = \lambda,\ x - y + z = 4$. Solving we get
$\lambda = 4/3, x = 5/3, y = 4/3, z = 11/3$, which is the point on the plane that has minimum distance from the given point.

4. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass. Use Lagrange multiplier method to find the dimensions of the aquarium that minimize the cost of the materials.

Solution: Hint volume $V = xyz,\ C(x,y,z) = 5xy + 2(xz + yz) =$ cost to minimize.
Answer: $x = y = \sqrt[3]{2/5V},\ z = \sqrt[3]{25/4V}$

5.