ARIZONA STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Chapter 13 Vector Functions

Section 13.1 Vector Functions and Space Curves

A vector valued function or simply a vector function is a function whose domain is a set of real numbers and whose range is a set of vectors.

Representation of a vector function: The vector $r(t)$ along with components $f(t), g(t), h(t)$, which are simple functions of t, represented as

$$r(t) =< f(t), g(t), h(t) >= f(t)i + g(t)j + h(t)k$$

Important properties:

1. $\lim_{t\to a} r(t) =< \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) >$, provided each limit exists

2. The parametric representations $x = f(t), y = g(t), z = h(t)$ is a space curve on I, t is a parameter.

Examples:

1. Find the domain of $r(t) =< t^3, \ln(2-t), \sqrt{t} >$

Solution: For, t^3, t is in I, set of real numbers. For $\ln(2-t)$, $t < 2$ and for $\sqrt{t}$, $t \geq 0$. Thus the domain is [0, 2)

2. Determine $\lim_{t\to 0} r(t)$ where $r(t) = \left\langle 1+\sqrt[3]{t}, \frac{1}{t-1}, \frac{\sin t}{t} \right\rangle$

Solution: $\lim_{t\to 0} r(t) =< 1, -1, 1 >$

3. Describe the curve defined by $r(t) =< 1+t, 2+5t, -1+6t >$

Solution: The parametric form of the curve is $x = 1+t, y = 2+5t, z = -1+6t$, which represents a straight line thru (1, 2, -1) in the direction of the vector $< 1, 5, 6 >$

4. Sketch the curve $r(t) =< \cos t, \sin t, t >$

Solution: The parametric form of the curve is $x = \cos t, y = \sin t, z = t$, which represents a curve that spirals around a circular cylinder with level curves $x^2 + y^2 = 1$. The curve is known as circular helix. See the figure at page # 851 in you text.

5. Find the vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$

Solution: Consider $x = \cos t, y = \sin t$, then $z = 2 - \sin t$, $0 \le t \le 2\pi$.
Now $r(t) = <\cos t, \sin t, 2 - \sin t> = i\cos t + j\sin t + k(2 - \sin t)$

6. Find the vector equation of a line thru the points P(1, 0, 1) and Q(2, 3, 1)

Solution: Use the formula thru two given points as $r(t) = (1-t)r_0 + tr_1,\ 0 \le t \le 1$
We have $r(t) = (1-t) <1,0,1> + t <2,3,1>,\ 0 \le t \le 1$

7. Identify the curve $r(t) = <e^{-t}\cos 10t, e^{-t}\sin 10t, e^{-t}>$

The curve is a spiral around a cone whose level curves are circles $x^2 + y^2 = e^{-t}$.
Observe that $x^2 + y^2 = z^2$

Section 13.2 Derivatives and Integrals of Vector Functions

For given vector function $r(t) = <f(t), g(t), h(t)> = f(t)i + g(t)j + h(t)k$
The derivative is defined as $r'(t) = <f'(t), g'(t), h'(t)> = f'(t)i + g'(t)j + h'(t)k$ and the integral is defined as $\int r(t)dt = <\int f(t)dt, \int g(t)dt, \int h(t)dt>$

Smooth curve: A curve given by a vector function $r(t)$ on an interval I is called smooth if $r'(t)$ is continuous and $r'(t) \ne 0$, where 0 is a zero vector.

Examples:

1. Given $r(t) = \left\langle 1 + \sqrt[3]{t}, \frac{1}{t-1}, \sin t \right\rangle$, find $r'(t)$ and $\int r(t)dt$

Solution: $r'(t) = \left\langle \frac{1}{3}t^{-2/3}, -\frac{1}{(t-1)^2}, \cos t \right\rangle$ and

$$\int r(t)dt = \left\langle t + \frac{3}{4}t^{4/3} + c_1, \ln(t-1) + c_2, -\cos t + c_3 \right\rangle$$

2. Find a unit tangent vector at (1, 0, 0) to the vector $r(t) = <1 + t^3, te^{-t}, \sin 2t>$

Solution: We have $r'(t) =< 3t^2, e^{-t} - te^{-t}, 2\cos 2t >$ and $r'(0) =< 0,1,2 >$, $\left|r'(0)\right| = \frac{1}{\sqrt{5}}$.

The unit tangent vector at $t = 0$ is $T(0) = \frac{r'(0)}{\left|r'(0)\right|} = \frac{1}{\sqrt{5}} < 0,1,2 >= \frac{1}{\sqrt{5}} j + \frac{2}{\sqrt{5}} k$

3. Find parametric equation for the tangent line to the helix with parametric equations $x = 2\cos t, y = \sin t$, and $z = t$ at $P(0,1,\pi/2)$

Solution: We have $r(t) =< 2\cos t, \sin t, t >$,
$r'(t) =< -2\sin t, \cos t, 1 > \Rightarrow r'(\pi/2) =< -2,0,1 >$. The tangent line passes thru $P(0,1,\pi/2)$ in the direction of $r'(\pi/2) =< -2,0,1 >$ having parametric equation
$x = -2t, y = 1, z = \pi/2 + t$

4. Find the tangent to the helix with parametric equations $x = 2\cos t, y = \sin t$, and $z = t$ at $P(0,1,\pi/2)$

Solution: Tangent at the given point is

$$T(\pi/2) = \frac{r'(\pi/2)}{\left|r'(\pi/2)\right|} = \frac{1}{\sqrt{5}} < -2,0,1 >$$

We have $r(t) =< 2\cos t, \sin t, t >$, $r'(t) =< -2\sin t, \cos t, 1 > \Rightarrow r'(\pi/2) =< -2,0,1 >$ and $\left|r'(\pi/2)\right| = \frac{1}{\sqrt{5}}$

5. Determine whether the semicubical parabola $r(t) =< 1 + t^3, t^2 >$ is smooth.

Solution: $r'(t) =< 3t^2, 2t >=< 0,0 >$ for $t = 0$. Thus it is not smooth.

6. Given $r(t) =< 2\cos t, \sin t, 2t >$, find $\int_0^{\pi/2} r(t)dt$

Solution: $\int_0^{\pi/2} r(t)dt = \left[2i\sin t - j\cos t + kt^2\right]_0^{\pi/2} = 2i + j + \pi^2/4k$

7. Find the point of intersection of the tangent lines to the curve $r(t) =< \sin \pi t, 2\sin \pi t, \cos \pi t >$ at the points where $t = 0$, and $t = 0.5$

Solution: $r'(t) =< \pi\cos \pi t, 2\pi\cos \pi t, -\pi\sin \pi t >$,
$r(0) =< 0,0,1 >$, $r(0.5) =< 1,2,0 >$ and $r'(0) =< \pi, 2\pi, 0 >$, $r'(0.5) =< 0,0,-\pi >$
The equation of the tangent line at $t = 0$ is

$$< x, y, z >=< 0,0,1 > + u < \pi, 2\pi, 0 >=< \pi u, 2u\pi, 1 > \text{ and}$$

Tangent at $t = 0.5$ is $< x, y, z >=< 1,2,0 > + v < 0,0,-\pi >=< 1,2,-\pi v >$. At the point of intersection $\pi u = 1$, $\pi v = -1 \Rightarrow u = 1/\pi, v = -1/\pi$. The point is (1, 2, 1)

Section 13.3 The Arc Length and Curvature

The arc length of a curve $r(t) = < f, g, h >$ from $t = a$ to $t = b$ is defined by

$$L = \int_a^b \sqrt{(f')^2 + (g')^2 + (h')^2}\, dt = \int_a^b |r'(t)| dt .$$

Suppose $s(t)$ is the length of the part of C between $r(a)$ and $r(t)$ then

$$s(t) = \int_a^t \sqrt{(f')^2 + (g')^2 + (h')^2}\, du = \int_a^t |r'(u)| du$$

Curvature: If C is a smooth curve defined by $r(t) = < f, g, h >$, then $r'(t) \neq 0$, the unit vector $T(t) = \frac{r'(t)}{|r'(t)|}$. The curvature of a curve C at a given point is a measure of how quickly the curve changes direction at that point. The curvature is defined as $\kappa = \left|\frac{dT}{ds}\right|$, where T is the unit tangent vector and s is the arc length function. Remember that

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt}, \text{ then } \kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \left|\frac{T'(t)}{r'(t)}\right|$$

For a function defined as $y = f(x)$, $\kappa(x) = \frac{|y''|}{\sqrt{[1 + (y')^2]^3}}$

Theorem: The curvature of the curve given by the vector function $r(t) = < f, g, h >$ is

$$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

The Normal and Binormal: The vectors N and B defined below are called the normal and Binormal vectors,

$$N(t) = \frac{T'(t)}{|T'(t)|} \text{ and } B(t) = T(t) \times N(t)$$

The plane determined by the normal and binormal vectors N and B at a point P on the curve C is called the **normal plane** of C at P. It consists all lines that are orthogonal to the tangent vector T.

The pane determined by the vectors T and N is called the **osculating plane** of C at P.

The circle that lies in the osculating plane of C at P has the same tangent as C at P lies on the concave side of C towards N and has radius $\rho = \frac{1}{\kappa}$, is called the osculating plane.

Examples:

1. Find the length of the arc of the circular helix with vector equation $r(t) = i\cos t + j\sin t + kt$ from the point (1, 0, 0) to $(1, 0, 2\pi)$

Solution: $L = \int_0^{2\pi} |r'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$

Where $|r'(t)| = |< -\sin t, \cos t, 1 >| = \sqrt{1+1} = \sqrt{2}$

2. Reparametrize the helix $r(t) = i\cos t + j\sin t + kt$ with respect to arc length s measured from (1, 0, 0) in the direction of increasing t.

Solution: We have $\frac{ds}{dt} = |r'(t)| = \sqrt{2}$ from example 1. Now

$s = \int_0^t \sqrt{2} dt = \sqrt{2}t \Rightarrow t = \frac{s}{\sqrt{2}}$ and $r(t(s)) = i\cos(s/\sqrt{2}) + j\sin(s/\sqrt{2}) + k(s/\sqrt{2})$

3. Show that the curvature of a circle of radius *r* is 1/*r*.

Solution: Consider $u(t) =< r\cos t, r\sin t >$ a circle of radius *r*. Now $\kappa = \left|\frac{T'(t)}{u'(t)}\right|$ and

$T(t) = \frac{u'(t)}{|u'(t)|} = \frac{< -r\sin t, r\cos t >}{r} =< -\sin t, \cos t >$, $T'(t) =< -\cos t, -\sin t >$, $|T'(t)| = 1$

The curvature of the circle of radius *r* is $\kappa = \left|\frac{T'(t)}{u'(t)}\right| = \frac{1}{r}$

4. Find the unit normal vector and binormal vector for the circular helix $r(t) =< \cos t, \sin t, t >$

Solution: $r'(t) =< -\sin t, \cos t, 1 >$, $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}} < -\sin t, \cos t, 1 >$

And $N(t) = \frac{T'(t)}{|T'(t)|} =< -\cos t, -\sin t, 0 >$,

$$B(t) = T \times N = \frac{1}{\sqrt{2}} \begin{pmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} < \sin t, -\cos t, 1 >$$

5. Find the equation of a normal plane and a osculating plane for the circular helix $r(t) =< \cos t, \sin t, t >$ at the point $P(0, 1, \pi/2)$

Solution: The normal plane has normal vector $r'(\pi/2) =< -1, 0, 1 >$.
The equation of the plane thru P is $-(x-0) + 0(y-1) + 1(z - \pi/2) = 0 \Rightarrow z = x + \pi/2$.

The osculating plane at P contains the vectors T and N. So from example 4, its normal vector is $B = T \times N = \frac{1}{\sqrt{2}} < \sin t, -\cos t, 1 >, B(\pi/2) = \frac{1}{\sqrt{2}} < 1,0,1 >$. The equation of the osculating plane is $\frac{1}{\sqrt{2}}(x-0) + \frac{1}{\sqrt{2}}(z - \pi/2) = 0 \Rightarrow z = -x + \pi/2$

6. Find the curvature of the parabola $y = x^2 + 1$ at the point (1, 2)

Solution: The curvature is $\kappa(x) = \frac{|y''|}{\sqrt{[1+(y')^2]^3}} = \frac{2}{(1+4x^2)^{3/2}}$.

At $x = 1$, $\kappa(1) = \frac{2}{(1+4)^{3/2}} \approx 0.18$

7. Find the osculating circle of $y = x^2$ at (0, 0)

Solution: $\kappa(x) = \frac{|y''|}{\sqrt{[1+(y')^2]^3}} = \frac{2}{(1+4x^2)^{3/2}} = 2,\ x = 0$. The radius of the osculating circle is $\rho = 1/2$, center at (0, ½). The osculating circle has the equation
$x^2 + (y - 1/2)^2 = 1/4$

8. Find curvature of $r(t) =< e^t \cos t, e^t \sin t, t >$ at $t = 0$.

Solution: $\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$r'(t) =< e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 1 >$, $r'(0) =< 1,1,1 >$

$r''(t) =< 2e^t \sin t, 2e^t \cos t, 0 >$, $r''(0) =< 0,2,0 >$

Check that $r'(0) \times r''(0) =< -2,0,2 >$

Then $\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{2\sqrt{2}}{3\sqrt{3}}$

9. Given $r(t) =< e^t, e^t \sin t, e^t \cos t >$. Find N, T, B at (1, 0, 1)

Solution: $T(t) = \frac{r'(t)}{|r'(t)|}$, $N(t) = \frac{T'(t)}{|T'(t)|}$ and $B(t) = T(t) \times N(t)$

$r'(t) = e^t < 1, \sin t + \cos t, \cos t - \sin t >$

When $e^t = 1 \Rightarrow t = 0$, $r'(0) =< 1,1,1 >$, $|r'(0)| = \sqrt{3}$ and

$T(t) = \frac{r'(t)}{|r'(t)|} = e^t < 1, \sin t + \cos t, \cos t - \sin t >$

$T'(t) = < 0, \cos t - \sin t, -\sin t - \cos t >$

$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{1}{\sqrt{3}} < 1,1,1 >,\ N(0) = \frac{T'(0)}{|T'(0)|} = \frac{1}{\sqrt{2}} < 0,1,-1 >$

$B(0) = T(0) \times N(0) = \frac{1}{\sqrt{6}} < -2,1,1 >$

Section 13.4 Motion in Space: Velocity and Acceleration

Suppose a particle moves thru the space so that its position vector at time *t* is *r*(*t*). The quantity

$$\frac{r(t+h) - r(t)}{h}$$

approximates the direction of the particle moving along the curve *r*(*t*). Its magnitude measures the size of the displacement vector per unit time. Thus the quantity we have is a vector that gives the average velocity over a time interval of length *h* and its limit is the velocity vector *v*(*t*) at time *t* defined as

$$v(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} = r'(t)$$

The speed of the particle at time t is the magnitude of the velocity, that is

$$\text{speed} = |v(t)| = \left| \frac{r(t+h) - r(t)}{h} \right| = |r'(t)| = V$$

As in the case of one-dimensional motion, the acceleration of the particle is defined as the derivative of the velocity:

$$a(t) = v'(t) = r''(t)$$

Tangent and normal components of acceleration: We have the tangent vector

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|} = \frac{v(t)}{\mathrm{V}} \Rightarrow v = VT$$

Differentiating velocity vector we have acceleration $a = v' = \mathrm{v}'T + VT'$

The curvature is then defined as $\kappa = \frac{|T'|}{|r'|} = \frac{|T'|}{\mathrm{V}} \Rightarrow |T'| = \kappa V$

The unit normal vector is $N = \frac{T'}{|T'|} = \frac{T'}{\kappa V} \Rightarrow T' = N\kappa V$

Thus we have $a = v' = V'T + VT' = V'T + \kappa V^2 N = a_T T + a_N N$, where $a_T = \mathrm{V}' = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$ is the tangential component and $a_N = \kappa V^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$ is the normal component of the acceleration.

Examples:

1. Find the velocity, acceleration and speed of a particle with position function $r(t) = < t^2 + 1, t^3, t^2 - 1 >$

Solution: Velocity $v(t) = r'(t) = < 2t, 3t^2, 2t >$

Acceleration $a(t) = v'(t) = r''(t) = < 2, 6t, 2 >$

Speed $V = |r'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2}$

2. Find the velocity, acceleration and speed of a particle with position function $r(t) = t^2\mathrm{i} + t^3\mathrm{j} + t^2\mathrm{k}$

Solution: Velocity $v(t) = r'(t) = 2t\mathrm{i} + 3t^2\mathrm{j} + 2t\mathrm{k}$

Acceleration $a(t) = v'(t) = r''(t) = 2\mathrm{i} + 6t\mathrm{j} + 2\mathrm{k}$

Speed $V = |r'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2}$

3. Given the acceleration vector $a(t) = \mathrm{i} + 2\mathrm{j} + 2t\mathrm{k}$, find velocity vector and position vector when $v(0) = 0, r(0) = \mathrm{i} + \mathrm{k}$

Solution: $a(t) = v'(t) = \frac{dv}{dt} = \mathrm{i} + 2\mathrm{j} + 2t\mathrm{k}$. Now integrating we have velocity $v(t) = t\mathrm{i} + t^2\mathrm{j} + t^2\mathrm{k} + \mathrm{c}_1$, using condition $v(0) = 0, \mathrm{c}_1 = 0$.

Again, $v(t) = r'(t) = \frac{dr}{dt} = t\mathrm{i} + t^2\mathrm{j} + t^2\mathrm{k}$, integrating we have position $r(t) = \frac{1}{2}t^2\mathrm{i} + \frac{1}{3}t^3\mathrm{j} + \frac{1}{3}t^3\mathrm{k} + \mathrm{c}_2$, using initial condition $r(0) = \mathrm{i} + \mathrm{k}$ we find $\mathrm{c}_2 = \mathrm{i} + \mathrm{k}$

4. Find the tangential and normal components of the acceleration vector $r(t) = t\mathrm{i} + t^2\mathrm{j} + 3t\mathrm{k}$

Solution: tangential component $a_T = \mathrm{V}' = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$ and the normal component is $a_N = \kappa V^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$. You can find them now. See also example 7 in your text at page number 875.