Arizona State University
DEPARTMENT OF MATHEMATICS AND STATISTICS

Chapter 12 Vectors and Geometry of Space

Section 12.1 Three-Dimensional Coordinate System

Suppose $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are the given points. Find the distance D. The distance D = $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

Important equations in 3-D to remember:
1. $ax+by+cz=d$ represents a plane
2. $x=a$ is a surface parallel to yz-plane
3. $y=b$ is a surface parallel to zx-plane
4. $z=c$ is a surface parallel to xy-plane
5. $y=x$ is a vertical plane that intersects xy-plane in the line $y=x$
6. $(x-a)^2+(y-b)^2+(z-c)^2=d^2$ is a sphere center at (a,b,c) and radius d
7. $(x-a)^2+(y-b)^2+(z-c)^2=d^2$, $z\ge c$ is a hemisphere center at (a,b,c)

Examples:

1. Find the equation of a sphere center at $(1,2,-1)$ and radius 1.

Solution: $(x-1)^2+(y-2)^2+(z+1)^2=1$

2. Determine whether the points lie on the straight line

a) $A(5,1,3),\ B(7,9-1), C(1,-15,11)$

Solution: Check that $AB=2\sqrt{21},\ BC=6\sqrt{21},\ AC=4\sqrt{21}$ and $AB+AC=BC$, The points are on a line.

b) $K(0,3,-4),\ L(1,2,-2), C(3,0,1)$. Like in a) you can show the points are not on the same line.

3. Find the center and radius of the sphere given by $x^2+y^2+z^2-6x+4y-2z=11$

Solution: Complete the square as $(x-3)^2+(y+2)^2+(z-1)^2=5^2$ and then center is at $(3,-2,1)$ and radius 5.

4. Describe in words the region of $\Re^3$ represented by the equations or inequalities

a) $y=-5$ b) $x=5$ c) $x>4$ d) $y\ge 0$
e) $0\le z\le 6$ f) $y=z$ g) $x^2+y^2+z^2>1$ h) $xyz=0$
i) $x^2+y^2+z^2-2z<3$ j) $x^2+y^2=1$ k) $x^2+z^2\le 9$

Section 12.2 Vectors

Parallelogram law: If we place two vectors $\vec{u},\vec{v}$ so that they start at a same point, then $\vec{u}+\vec{v}$ lies along the diagonal of the parallelogram with $\vec{u},\vec{v}$ vectors as sides.

For two vectors $\vec{u} =< a,b,c >$ and $\vec{v} =< x,y,z >$ the vector represented and defined by $\vec{a} = \overrightarrow{AB} =< x-a, y-b, z-c >$ and $-\vec{a} = \overrightarrow{BA} =< a-x, b-y, c-z >$
The length or magnitude of a vector: $|\vec{a}| = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

Examples:

1. Given $\vec{a} =< 3,-1 >, \vec{b} =< 5,3 >$. Find $\vec{a}+\vec{b}$, $\vec{a}-2\vec{b}$, $3\vec{a}-\vec{b}$ and $|\vec{a}+\vec{b}|$, $|\vec{a}-2\vec{b}|$, $|3\vec{a}-\vec{b}|$
2. Find a vector that has same direction as the vector $<-2,4,5>$ and magnitude 6.

Solution: find unit vector $\vec{u} = \frac{1}{3\sqrt{5}} <-2,4,5>$, the vector we are looking for is

$\vec{w} = 6\vec{u} = \frac{2}{\sqrt{5}} <-2,4,5>$

Section 12.3 The Dot Product of Vectors

For two vectors $\vec{u} =< a,b,c >, \vec{v} =< x,y,z >$ the dot product is defined as $\vec{u}\cdot\vec{v} = ax+by+cz$

If θ is the angle between the vectors $\vec{u},\vec{v}$, then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Case 1. $\theta = 0$ vectors are parallel in the same direction and $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$

Case 2. $\theta = -\pi$ vectors are parallel in the opposite direction and $\vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$

Case 3. $\theta = \frac{\pi}{2}$ vectors are perpendicular (Orthogonal) and $\vec{u}\cdot\vec{v} = 0$

Direction angles and direction cosine: If α, β, γ are the angles of a vector with the coordinate axes x, y and z respectively then those are called the direction angles. And $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines and $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$

Question: Show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$

Projections: Prove that scalar projection of $\vec{b}$ onto $\vec{a}$ is $comp_a b = \frac{\vec{a}\bullet\vec{b}}{|a|}$ and

Vector projection of $\vec{b}$ onto $\vec{a}$ is $proj_a b = \left(\frac{\vec{a}\bullet\vec{b}}{|a|}\right)\frac{\vec{a}}{|a|}$ (See your text book page # 811)

Examples:

1. Find the dot product between two given vectors.

   a) $\vec{u} = \left\langle \frac{1}{2}, 4 \right\rangle$ and $\vec{v} = < -8, -3 >$ b) $\vec{u} = 4i - 3k$ and $\vec{v} = 2i + 3j - 4k$

2. Given that $|\vec{u}| = 4,$ and $|\vec{v}| = 10,\ \theta = 120^0$ find $\vec{u}\cdot\vec{v}$

3. Find the angle between the vectors $\vec{u} = \left\langle \frac{1}{2}, 4 \right\rangle$ and $\vec{v} = < -8, -3 >$

4. Show that $\vec{u} = < 2, 6, -4 >$ and $\vec{v} = < -3, -9, 6 >$ are parallel

5. Find a unit vector that is orthogonal to both $i + j$ and $i + k$

   Solution: Suppose $\vec{a} = < a, b, c >$ is the unit vector. Now $a^2 + b^2 + c^2 = 1$ and $< a, b, c > \bullet < 1, 1, 0 > = 0$ and $< a, b, c > \bullet < 1, 0, 1 > = 0$. Solve for *a*, *b*, and *c* for the unit vector $\vec{a} = < a, b, c > = \pm\frac{1}{\sqrt{3}} < 1, -1, -1 >$.

6. Find direction cosines and direction angles of the vector $\vec{a} = < 2, 3, -6 >$

Solution: $\cos\alpha = \frac{2}{\sqrt{4+9+36}} = \frac{2}{7} \Rightarrow \alpha = \arccos(2/7)$,

$\cos\beta = \frac{3}{\sqrt{4+9+36}} = \frac{3}{7} \Rightarrow \beta = \arccos(3/7)$ and

$\cos\gamma = \frac{-6}{\sqrt{4+9+36}} = \frac{-6}{7} \Rightarrow \gamma = \arccos(-6/7)$

7. If two direction angles are given $\alpha = \pi/4,\ \beta = \pi/3$, find γ.

8. Determine the scalar and vector projection of $\vec{b} = < -4, 1 >$ onto $\vec{a} = < 1, 2 >$

9. A crate is hauled 8 cm up a ramp under a constant force of 300 N applied at an angle of 30 degrees to the ramp. Find the work done.

   Solution: Work done W = $F\cdot D = |F||D|\cos\theta = 300(8)\cos 30^0\, Nm = 2078.46J$
   Nm stands for Newton-meter and 1 Nm = 1 Joules, which is the unit of work.

10. A force is given by a vector F = <3, 4, 5> and moves a particle from the point P(2,1, 0) to the point Q(3, 2, 1). Find the work done.

Solution: $F \cdot D = <3,4,5> \cdot <1,1,1>$ units = 12 units of work, where D is the distance vector from P to Q.

11. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 20 ft long and inclined at an angle of 15 degrees above the horizontal. Find the work done on the box. Answer: About 483 ft-lb

Section 12.4 The Cross Product of Vectors

For two vectors $\vec{u} = <a,b,c>, \vec{v} = <d,e,f>$ the cross product is defined as

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i(bf - ec) - j(af - dc) + k(ae - bd)$$

You need to review determinants from any algebra book.

Theorem: The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
Theorem: If θ is the angle between two vectors then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta,\ 0 \le \theta \le \pi$
Corollary: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = \vec{0}$

Note: The length of the cross product $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram determined by the vectors.

Scalar Triple Product (STP): For three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ the scalar triple product is defined as $\vec{u} \cdot (\vec{v} \times \vec{w})$

Volume of a parallelepiped is given by $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$, which is the magnitude of a scalar triple product.

Note: If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ the vectors are coplanar

Properties:
1. $\vec{a} \cdot (\vec{b} \times \vec{c}) = c \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$
2. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
3. $i \times i = 0$, $i \times j = k$, $k \times i = j$ and so on
4. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b}) - (\vec{a} \cdot \vec{b})\vec{c}$

Examples:
1. Show that $\vec{a} = <1,4,-7>, \vec{b} = <2,-1,4>$ and $\vec{c} = <0,-9,18>$ are coplanar

Solution: One needs to verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

2. Find a vector perpendicular to the plane that that passes thru $P(1,4,6), Q(-2,5,-1)$ and $R(1,-1,1)$

Solution: Find $\overrightarrow{PQ}\times\overrightarrow{PR} = <-40,-15,15>$

3. Find the area of a triangle with vertices $P(1,4,6), Q(-2,5,-1)$ and $R(1,-1,1)$

Solution: Area $A = 1/2\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \dfrac{5\sqrt{82}}{2}$

4. Find $\vec{a}\times\vec{b}$ for $\vec{a} = <1,4,-7>, \vec{b} = <2,-1,4>$ and show that the cross product is orthogonal to both $\vec{a}$ and $\vec{b}$
5. Find two unit vectors orthogonal to both $<1,1,1>$ and $<2,0,1>$
6. Show that $\vec{a}\cdot(\vec{b}\times\vec{a}) = 0$
7. Show that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix}$

Solution:

$$
\begin{aligned}
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\times\vec{b})\cdot\vec{v} = \vec{a}\cdot(\vec{b}\times\vec{v}) \\
&= \vec{a}\cdot(\vec{b}\times\vec{c}\times\vec{d}) = \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}] \\
&= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})
\end{aligned}
$$

Section 12.5 Equations of lines and planes

Lines: For a given direction vectors $\vec{v} = <a,b,c>$, the vectors $\vec{r} = <x,y,z>$, and $\vec{r}_0 = <x_0, y_0, z_0>$ lies on the line L

$$
\begin{aligned}
\vec{r} &= \vec{r}_0 + t\vec{v} \\
<x,y,z> &= <x_0,y_0,z_0> + t<a,b,c> \\
&= <x_0 + at, y_0 + bt, z_0 + ct>
\end{aligned}
$$

The parametric equation of the line L is $x = x_0 + at, \ \ y = y_0 + bt, z = z_0 + ct$

Also we can write the symmetric form (eliminating t)

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where *a, b*, and *c* are called direction numbers or direction ratios.

The line segment from $\vec{r}_0 = <x_0, y_0, z_0>$ to $\vec{r}_1 = <x_1, y_1, z_1>$ is the vector given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \ 0 \le t \le 1$$

For two points $p_0 = (x_0, y_0, z_0)$ and $p_1 = (x_1, y_1, z_1)$ on L has the symmetric equation

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Planes: Vector equation of a plane is defined as $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$, where $\vec{n}$ is the unit normal vector to the plane, containing $\vec{r}$, and $\vec{r}_0$.

A plane passing thru a point $p_0=(x_0,y_0,z_0)$ with normal vector $\vec{n}=<a,b,c>$ has scalar equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ which also can be written as $ax+by+cz+d=0$. The distance of a point $p_1=(x_1,y_1,z_1)$ from the plane $ax+by+cz+d=0$ is defined as $D=\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Angle between two planes $ax+by+cz+d=0$ and $a_1x+b_1y+c_1z+d_1=0$ is given by $\cos\theta=\dfrac{n_1\cdot n_2}{|n_1||n_2|}$, where $n_1=<a,b,c>$ and $n_2=<a_1,b_1,c_1>$

Examples:

A. Lines

1. Find a vector equation and parametric equations for the line that passes thru (5, 1, 3) and is parallel to $\vec{v}=<1,4,-2>$

   Vector equation
   $$<x,y,z>=<5,1,3>+t<1,4,-2>$$
   $$=<5+t,1+4t,3-2t>=(5+t)i+(1+4t)j+(3-2t)k$$

   Parametric equation $x=5+t,\ y=1+4t,\ z=3-2t$

2. Find a symmetric equation and parametric equations for the line that passes thru (2, 4, 3) and (3, 1, 1)

   The symmetric equation
   $$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}$$
   $$\frac{x-2}{1}=\frac{y-4}{-3}=\frac{z-3}{-2}$$
   The parametric form is $x=2+t,\ y=4-3t,\ z=3-2t$

3. In example 2, find intersection of the line with xy-plane.

   On the xy-plane $z=0$. Then $x=2+t,\ y=4-3t,\ z=0=3-2t\Rightarrow t=3/2$
   We have the point (7/2, -1/2, 0)

B. Planes

4. Find an equation of a plane through (2, 4, -1) with a normal vector $\vec{n} = <2,3,4>$

The plane has equation
$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0 \Rightarrow <2,3,4>\cdot<x-2, y-4, z+1>=0 \Rightarrow 2(x-2)+3(y-4)+4(z+1)=0$

5. Find the equation of a plane thru P(1, 3, 2), Q(3, -1, 6) and R(5, 2, 0)

Derive vectors $\overrightarrow{PQ} = <2,-4,4>$, $\overrightarrow{PR} = <4,-1,-2>$, and $\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR}$. Now you can consider the point P(1, 3, 2) and the normal vector to find your plane
$12(x-1)+20(y-3)+14(z-2)=0$

6. Find the angle between two given planes $x+y+z=1$ and $x-2y+3z=1$.

Notice that we have $\vec{n}_1 = <1,1,1>$ and $\vec{n}_2 = <1,-2,3>$. Now find $\cos\theta = \frac{n_1\cdot n_2}{|n_1||n_2|}$

7. Find the symmetric equations of the line of intersection L of two planes $x+y+z=1$ and $x-2y+3z=1$.

Suppose $\vec{n}_1$ and $\vec{n}_2$ are the normal vectors to the given planes. Then $n_1 = <1,1,1>$ and $n_2 = <1,-2,3>$. The line L has direction vector $v = n_1\times n_2 = <5,-2,-3>$. Let us find a point common to both the planes letting z = 0, which could be (1, 0, 0). Thus we have the equation of L in symmetric form, $\frac{x-1}{5}=\frac{y}{-2}=\frac{z}{-3}$

Section 12.6 Cylinder and quadric surfaces

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and passes thru a given plane curve.

A quadric surface is the graph of a second-degree equation in three variables *x*, *y* and *z*. The most general equation of a quadric surface is
$Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+x+Hy+Iz+J=0$

General forms:
Look at page number 386 on your text for the diagrams

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is an ellipsoid. For $a=b=c$, the ellipsoid is a sphere
2. $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ is an elliptic paraboloid. For $a=b$ it is circular paraboloid.

3. $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ is a hyperbolic paraboloid.
4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is a hyperboloid of one sheet
5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ is a hyperboloid of two sheets
6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ is acone

Homework problems:

8. since z is missing in $x^2-y^2=1$, we consider $x^2-y^2=1$ with $z = k$, is a hyperbola on the $z = k$ plane. The surface is hyperbolic cylinder.

12. Find the traces of $4y=x^2+z^2$ in the planes $x = k$, $y = k$, and $z = k$.

When $x = k$: $4y=k^2+z^2$ is a parabola,

$y = k$: $4k=x^2+z^2$ is a circle

and $z = k$: $4y=x^2+k^2$ is also a parabola

Thus the surface is a circular paraboloid with axis in the y axis and vertex (0, 0, 0)

22. $9x^2+4y^2+z^2=1\Rightarrow\frac{x^2}{1/9}+\frac{y^2}{1/4}+\frac{z^2}{1}=1$ is an ellipsoid with intercepts $(\pm1/\sqrt{3},0,0)$, $(0,\pm1/2,0)$, $(0,0,\pm1)$

34. Reduce the equation $4y^2+z^2-x-16y-4z+20=0$ to one of the standard forms a classify the surface and make a rough sketch.

Solution: We find the form $\frac{x}{4}=\frac{(y-2)^2}{1}+\frac{(z-2)^2}{4}$ is an elliptic paraboloid vertex at (0, 2, 2) and axis is the horizontal line $y = 2$, $z = 2$.