Numerical Methods for Conservation Laws

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LITERATURE

1. Numerical Methods for Conservation Laws, by Randall J. LeVeque, Birkhauser-Verlag, Basel, 1990. ISBN 3-7643-2464-3

2. J. Strickwerda: Finite difference schemes and partial differential equations, Wadsworth and Brooks/Cole, 1992.

OUTLINE

6 lectures

Week 1: Mo 11:00-12:30, Tue 12:30-14:00 Wed 11:00-12:30

Week 2: Mo 11:00-12:30, Tue 12:30-14:00 Wed 12:30-14:00

Materials: http://math.la.asu.edu/ chris $\rightarrow$ Courses

Topics:

L1 Conservation laws and conservative formulation, Difference methods, Linear waves, Diffusion, Dispersion, CFL conditions. Examples: Wave equations, diffusion equations, Schröedinger equations.

L2 Nonlinear equations, Characteristics, Shocks, Weak solutions, The Rankine - Hugoniot condition. Examples: Traffic Flow.

L3 Godunov methods, Local Riemann problems, Shock capturing, Systems, local linearization and Roe matrices. Examples: Traffic flow, Fluids, Compressible Euler.

L4 Higher order methods, TVD schemes.

L5 Convergence analysis, Stability and compactness.

L6 Topics: Random methods, Entropy, Boundary conditions. Examples: Boltzmann equations.

LECTURE 1

Conservation Laws

$U_\Omega(t)$: amount of 'mass' in Ω at time t
F : flux through the boundary $\partial\Omega$

$$U_\Omega(t+\Delta t) = U_\Omega(t) - \Delta t \int_{\partial\Omega} F \cdot \mathbf{n} \, d\sigma = U_\Omega(t) - \Delta t \int_\Omega \nabla_x \cdot F \, dx$$

Size of $|\Omega|$ and $\Delta t \to 0$:

$$\partial_t u + \nabla_x \cdot F = 0, \quad U_\Omega(t) = \int_\Omega u(x,t) \, dx$$

Examples: $F = b(u)u$ (transport), $F = -a(u)\nabla_x u$ (diffusion)

Wave speeds

Linear problems and plane wave solutions

$$F(u) = bu - a\nabla_x u, \quad u(x,t) = \exp[i\xi(x - vt)]$$

v : velocity, $v\xi$: frequency

$$u(x,t) = \exp[i\xi(x - bt) - a|\xi|^2 t]$$

wave speed $= b$ (property of the medium). amplitude damped with $\exp[-a|\xi|^2 t]$.

Dispersion:

$$\partial_t u + i\Delta_x u = 0, \quad v = v(\xi) = \xi$$

wave speed dependent on the frequency!

hyperbolicity $\iff$ v only medium dependent
dispersivity $\iff$ v frequency dependent
parabolicity $\iff$ exponential damping

Systems: $\partial_t u + \partial_{x_j}[A_j u]$

Difference methods always exhibit all three features, regardless of the type of equation!

Difference methods for advection diffusion equations

A linear example:

$$\partial_t u + \partial_x[-a\partial_x u + bu] = 0, \quad u(x,0) = u^I(x)$$

Discretization: Grid: $x_j = jh, \quad j \in \mathbb{Z}$, $t_n = nk, n \in \mathbb{N}$, approximate $u(x_j,t_n) \approx U(x_j,t_n)$
Difference approximation:

$$U(x_j,t_{n+1}) = U(x_j,t_n) +$$

$$\frac{k}{h^2}[(a + \frac{h}{2}b)U(x_{j-1},t_n) - 2aU(x_j,t_n) + (a - \frac{h}{2}b)U(x_{j+1},t_n)]$$

$$U(x_j,t_0) = u^I(x_j)$$

advance in time: $U(*,t_n) \to U(*,t_{n+1})$, $t_{n+1} - t_n = k$

Notation: The translation operator:

$$Tu(x) = u(x+h), \quad T^{-1}u(x) = u(x-h)$$

$$U(x,t+k) = \{1 + \frac{k}{h^2}[(a+\frac{h}{2}b)T^{-1} - 2a + (a-\frac{h}{2}b)T]\}U(x,t)$$

Linear wave ansatz: $U(x,t) = \exp[i\xi(x-vt)]$

$$\exp[-i\xi vk] = 1 - a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2}) - ib\frac{k}{h}\sin(\xi h)$$

$e^{-i\xi v} = qe^{-i\xi v_r},\ \ v_r \in \mathbb{R}$

$$U(x,t) = q^t \exp[i\xi(x-v_r t)]$$

$$q^{2k} = [1-a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2})]^2 + [b\frac{k}{h}\sin(\xi h)]^2, \quad \tan(\xi k v_r) = \frac{b\frac{k}{h}\sin(\xi h)}{1-a\frac{4k}{h^2}\sin^2(\frac{\xi h}{2})}$$

Wave speeds always dependent on frequencies.

For $a = 0$ waves are always amplified (instability!)

Difference discretizations are always dispersive.

Explicit difference discretizations always need some amount of diffusion to be stable (artificial diffusion).

$a = k\alpha,\ c = \frac{k}{h},\ z = \sin(\frac{\xi h}{2})$

$$q^{2k} = [1 - 4\alpha c^2 z^1]^2 + b^2 c^2 z^2 (1 - z^2) \leq 1, \quad \forall z \in [0, 1]$$

$\Rightarrow b^2 c^2 \leq 2akc^2 < 1$

CFL condition: $\frac{k|b|}{h} < 1$

Artificial diffusion: $a = O(k)$

Also true for nonlinear problems and problems with non - constant coefficients!

LECTURE 2

CHARACTERISTICS AND SHOCKS

Characteristics for linear and scalar problems:

$$\partial_t u + a(x,t)\partial_x u = 0, \quad u(x,0) = u^I(x)$$

Characteristic:
Try to find a solution of the form $u(\xi(t),t) = const$

$$\frac{d}{dt}\xi_{ys}(t) = a(\xi_{ys},t), \quad \xi_{ys}(s) = y,$$

$$\Rightarrow \quad \frac{d}{dt}u(\xi_{ys}(t),t) = 0, \quad u(\xi_{ys}(t),t) = u(y,s)$$

$$\text{forward: } u(\xi_{y0}(t),t) = u^I(y),$$

backward: The solution $u(x,t)$ is given by

$$u(y,s) = u(\xi_{ys}(t),t) = u^I(\xi_{ys}(0)),$$

Conservative formulation:

$$\partial_t u + \partial_x[a(x,t)u] = 0, \quad u(x,0) = u^I(x)$$

Characteristic:

$$\frac{d}{dt}\xi_{ys}(t) = a(\xi_{ys},t), \quad \xi_{ys}(s) = y,$$

$$\Rightarrow \quad \frac{d}{dt}u(\xi_{ys}(t),t) = -u(\xi_{ys}(t),t)\partial_x a(\xi_{ys}(t),t),$$

$$\frac{d}{dt}\xi_{ys}(t) = a(\xi_{ys},t), \quad \frac{d}{dt}v_{ys}(t) = -v_{ys}\partial_x a(\xi_{ys},t),$$

$$v_{ys}(t) = u(\xi_{ys}(t),t), \quad \xi_{ys}(s) = y, \quad v_{ys}(s) = u(y,s),$$

forward: $\frac{d}{dt}\xi_{y0} = a(\xi_{y0},t), \quad \frac{d}{dt}v_{y0} = -v_{y0}\partial_x a(\xi_{y0},t),$

$$\xi_{y0}(0) = y, \quad v_{y0}(0) = u^I(y),$$

Characteristics can never intersect for linear problems!

Find the solution: Given (y, s)

backward solve:

$$\xi'_{ys}(t) = a(\xi_{ys}, t),\ \xi_{ys}(s) = y \ \rightarrow \ \xi_{ys}(0) := \xi^0_{ys}$$

forward solve:

$$v'_{ys}(t) = -\partial_x a(\xi_{ys}, t) v_{ys}, \quad v_{ys}(0) = u_I(\xi^0_{ys}) \ \rightarrow \ v_{ys}(s) = u(\xi_{ys}(s), s) = u(y$$

Example: $\partial_t u + \partial_x [xu] = 0$

Nonlinear problems:

$$\partial_t u + \partial_x f(x, t, u) = 0, \quad u(x, 0) = u^I(x)$$

$$\frac{d}{dt}\xi_{ys} = \partial_u f(\xi_{ys}, t, v_{ys}), \quad \frac{d}{dt} v_{ys} = -\partial_x f(\xi_{ys}, t, v_{ys}),$$

For nonlinear problems characteristics can intersect. The solution will develop a discontinuity (shock)!

Example: Burger's equation $\partial_t u + \partial_x \frac{u^2}{2} = 0, \quad u(x, 0) = u^I(x)$

THE CONCEPT OF WEAK SOLUTIONS

$$\partial_t u + \partial_x f(u) = 0, \quad u(x,0) = u^I(x) \tag{1}$$

u discontinuous $\Rightarrow$ need to re- define derivatives
ψ sufficiently smooth test function, $\psi \in C_0^\infty$

Definition: u is a weak solution of (1) iff

$$\int_0^\infty dt \int dx \; [u\partial_t\psi + f(u)\partial_x\psi] = -\int dx \; [\psi(x,0)u^I(x)] \; \forall\psi$$

Remark: u differentiable $\Rightarrow$ u classical solution.

Example:

$$\partial_t u + \partial_x[H(x)u + \frac{1}{2}H(-x)u] = 0, \quad u(x,0) = u^I(x)$$

THE RANKINE - HUGONIOT CONDITION

Relates shockheight to shockspeed if there is a shock.

$$\partial_t u + \partial_x f(u) = 0,$$

Assume:

1. u differentiable away from the shock curve $x = \gamma(t)$.
2. u is a weak solution.

Implies:

$$\gamma'(t)(u_+ - u_-)(\gamma(t), t) = (f_+ - f_-)(\gamma(t), t)$$

Example: Traffic flow

$$\partial_t u + \partial_x [c(1 - \frac{u}{u_0})u] = 0, \quad u(x,0) = u^I(x) = \begin{matrix} u_0 & for \quad x > 0 \\ \frac{1}{2}u_0 & for \quad x < 0 \end{matrix}$$

LECTURE 3

GODUNOV'S METHOD

Scalar problems:

$$\partial_t u + \partial_x f(u) = 0$$

Equation for averages:

$$U_j^n = \frac{1}{h}\int_{x_j}^{x_{j+1}} u(x,t_n)dx \ , \ F_j^n = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(u(x_j,t))dt \ .$$

$$\Rightarrow \ U_j^{n+1} - U_j^n + \frac{k}{h}(F_{j+1}^n - F_j^n) = 0$$

Up to here exact!

Numerical approximation: compute F_j^n from U_{j-1}^n, U_j^n

Piecewise constant approximation: Assume $u(x,t_n) = U_j^n$ for $x_j < x < x_{j+1}$.

Solve problem with piecewise constant initial data (the Riemann problem) exactly.

THE RIEMANN PROBLEM

$$\partial_t u + \partial_x f(u) = 0,\ u(x,0) = \begin{pmatrix} U_L & \text{for} & x < 0 \\ U_R & \text{for} & x > 0 \end{pmatrix}$$

Compute $F = \frac{1}{k}\int_0^k f(u(0,t))dt$.

Characteristics: $x' = f'(u),\ u' = 0$
Shock curve according to Rankine - Hugoniot:
$\gamma'(U_R - U_L) = f(U_R) - f(U_L)$

Case 1: $f'(U_R) > 0, f'(U_L) > 0 \ \Rightarrow\ F = f(U_L)$
Case 2: $f'(U_R) < 0, f'(U_L) < 0 \ \Rightarrow\ F = f(U_R)$
Case 3: $f'(U_R) < 0 < f'(U_L)$ and $\gamma' > 0 \ \Rightarrow\ F = f(U_L)$
Case 4: $f'(U_R) < 0 < f'(U_L)$ and $\gamma' < 0 \ \Rightarrow\ F = f(U_R)$
Case 5: $f'(U_L) < 0 < f'(U_R)$: rarefaction wave and the Bärenblatt solution.

THE BÄRENBLATT SOLUTION

$$\partial_t u + \partial_x f(u) = 0, \text{ set } u(x,t) = g(\frac{x}{t}) \;\Rightarrow\; f'(g(z)) = z$$

$u(x,t) = (f')^{-1}(\frac{x}{t})$ gives continuous solution in Case 5!

Case 5: $f'(U_L) < 0 < f'(U_R) \;\Rightarrow\; F = f(u_s)$ with $f'(u_s) = 0$.

u_s is called the sonic point and $u(x,t) = (f')^{-1}(\frac{x}{t})$ is called a rarefaction wave.

Godunov:

$$\gamma' := \frac{f(U_j^n) - f(U_{j-1}^n)}{U_j^n - U_{j-1}^n}$$

Case 1: $f'(U_j^n) > 0, f'(U_{j-1}^n) > 0 \;\Rightarrow\; F_j^n = f(U_{j-1}^n)$
Case 2: $f'(U_j^n) < 0, f'(U_{j-1}^n) < 0 \;\Rightarrow\; F_j^n = f(U_j^n)$
Case 3: $f'(U_j^n) < 0 < f'(U_{j-1}^n)$ and $\gamma' > 0 \;\Rightarrow\; F_j^n = f(U_{j-1}^n)$
Case 4: $f'(U_j^n) < 0 < f'(U_{j-1}^n)$ and $\gamma' < 0 \;\Rightarrow\; F_j^n = f(U_j^n)$
Case 5: $f'(U_{j-1}^n) < 0 < f'(U_j^n) \;\Rightarrow\; F_j^n = f(u_s)$ with $f'(u_s) = 0$.

SYSTEMS AND APPROXIMATE RIEMANN SOLVERS

$$\partial_t u + \partial_x f = 0, \qquad u \in \mathbb{R}^N$$

$$U_j^{n+1} - U_j^n + \frac{k}{h}(F_{j+1}^n - F_j^n) = 0, \qquad F_j^n = F(U_{j-1}^n, U_j^n, v_0),$$

Flux Approximation

Solution with approximate flux function $\tilde{f}_{U_L,U_R}$

$$\partial_t v + \partial_x \tilde{f}_{U_L,U_R}(v) = 0, \quad v_0 := v(x_j, t)$$

Requirements on F and $\tilde{f}_{U_L,U_R}$:
Conservation:

$$F(U_L, U_R, U_L) = f(U_L), \quad F(U_L, U_R, U_R) = f(U_R),$$

Shock speeds:

$$\tilde{f}_{U_L,U_R}(U_R) - \tilde{f}_{U_L,U_R}(U_L) = f(U_R) - f(U_L) \tag{2}$$

Set:

$$F(U_L, U_R, v) = \tilde{f}_{U_L,U_R}(v) + f(U_L) - \tilde{f}_{U_L,U_R}(U_L) \tag{3}$$

$$= \tilde{f}_{U_L,U_R}(v) + f(U_R) - \tilde{f}_{U_L,U_R}(U_R)$$

Step 1: Find $\tilde{f}_{U_L,U_R}$ satisfying (2)
Step2: Solve approximate conservation law for v_0.
Step3: Compute F according to (3).

Approximate Riemann solvers:

$$\partial_t v + \partial_x \tilde{f}_{U_L,U_R}(v) = 0, \quad v(x,0) = U_L,\ x < 0, \quad v(x,0) = U_R,\ x > 0$$

Linear approximation: $\tilde{f}_{U_L,U_R}(v) = A(U_L, U_R)v$
A : Roe matrix
Shock speeds:

$$A(U_L, U_R)(U_R - U_L) = f(U_R) - f(U_L)$$

diagonalize A and solve N scalar linear problems.

SOLVING THE CONSTANT COEFFICIENT RIEMANN PROBLEM FOR SYSTEMS

$$A(U_L, U_R)\tilde{r}_m = \tilde{\lambda}_m \tilde{r}_m, \quad Df(U_L) r_m^L = \lambda_m^L r_m^L, \quad Df(U_R) r_m^R = \lambda_m^R r_m^R$$

expand U_L, U_R in eigenvectors

$$U_L = \sum_m w_m^L \tilde{r}_m, \quad U_R = \sum_m w_m^R \tilde{r}_m,$$

solve

$$\partial_t w_m + \tilde{\lambda}_m \partial_x w_m = 0, \quad w_m(x,0) = \begin{pmatrix} w_m^L & \text{for} & x < 0 \\ w_m^R & \text{for} & x > 0 \end{pmatrix}$$

$$w_m(0,t) = \begin{pmatrix} w_m^L & \text{for} & \tilde{\lambda}_m > 0 \\ w_m^R & \text{for} & \tilde{\lambda}_m < 0 \end{pmatrix}, \quad v(0,t) = \sum_m w_m(0,t)\tilde{r}_m$$

Problem:

Only shock solutions, no rarefaction waves.

except: $\lambda_m^L < 0 < \lambda_m^R$ (rarefaction wave)

replace by two shocks: $w_m(0,t) = w^s$

conservation:

$$w_m^L(k\lambda_m^L + C) + kw_m^s(\lambda_m^R - \lambda_m^L) + w_m^R(C - k\lambda_m^R) = w_m^L(k\tilde{\lambda}_m + C) + w_m^R(C - k\tilde{\lambda}_m)$$

$$w_m^s = \frac{w_m^L(\tilde{\lambda}_m - \lambda_m^L) + w_m^R(\lambda_m^R - \tilde{\lambda}_m)}{\lambda_m^R - \lambda_m^L}$$

This is sometimes called the 'sonic fix'.

SUMMARY

Roe matrix:

$$A_j^n = A(U_{j-1}^n, U_j^n), \quad A_j^n(U_j^n - U_j^{n-1}) = f(U_j^n) - f(U_{j-1}^n)$$

$$F_j^n = f(U_{j-1}^n) + A_j^n(v^0 - U_{j-1}^n), \quad v^0 = \sum_m w_m^0 \tilde{r}_m$$

$$w_m^0 = \begin{pmatrix} w_m^L & \text{for} & \tilde{\lambda}_m > 0 \\ w_m^R & \text{for} & \tilde{\lambda}_m < 0 \end{pmatrix}, \quad U_{j-1}^n = \sum_m w_m^L \tilde{r}_m, \quad U_j^n = \sum_m w_m^R \tilde{r}_m$$

except for the sonic fix:

$$Df(U_j^n) r_{jm}^n = \lambda_{jm}^n r_{jm}^n, \quad m = 1, .., N$$

If $\lambda_{j-1,m}^n < 0 < \lambda_{j,m}^n$ then

$$w_m^0 = \frac{w_m^L(\tilde{\lambda}_m - \lambda_{j-1,m}^n) + w_m^R(\lambda_{j,m}^n - \tilde{\lambda}_m)}{\lambda_{j,m}^n - \lambda_{j-1,m}^n}$$

ONE WAY TO FIND A ROE MATRIX

$$A(U_L, U_R) = \int_0^1 Df(U_L + s(U_R - U_L))ds$$

Example 1: Burger's equation, $f(u) = \frac{1}{2}u^2$

Example 2: Isothermal flow

$$f(\rho, \phi) = \begin{pmatrix} \phi \\ \frac{\phi^2}{\rho} + \rho \end{pmatrix},$$

$$A(U_L, U_R) = \begin{pmatrix} 0 & 1 \\ 1 - v^2 & 2v \end{pmatrix}, \quad v = \frac{\sqrt{\frac{\phi_L^2}{\rho_L}} + \sqrt{\frac{\phi_R^2}{\rho_R}}}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad v_{LR} = \frac{\phi_{LR}}{\rho_{LR}}$$

LECTURE 4

ODE METHODS AND THEIR PDE COUNTERPARTS

ODE: $\frac{du}{dt} = f(u,t)$

Euler's method (explicit): $U(t+k) = U(t) + kf(U,t)$

Higher order methods I: Runge Kutta

Idea: Write in integral form

$$u(t+k) = u(t) + k\int_0^1 f(u, t+sk)\ ds$$

Use integration rule for the integral
Euler: $\int_0^1 f(u,t+sk)\ ds \approx f(U,t)$
Second order: $\int_0^1 f(u,t+sk)\ ds \approx f(U_{1/2}, t+k/2)$; compute

$u_{1/2}$ by intermediate Euler step

$$U_{1/2} = U(t) + \frac{k}{2} f(U,t), \quad U(t+k) = U(t) + k f(U_{1/2}, t + \frac{k}{2})$$

Equivalent for hyperbolic PDE's: Lax-Wendroff in staggered grid formulation

$$\partial_t u + \partial_x f(u) = 0, \quad U(t+k) = U(t) - \frac{k}{h}(T^{1/2} - T^{-1/2}) f(U(t + \frac{k}{2}))$$

$$U(t + \frac{k}{2}) = \frac{1}{2}(T^{1/2} + T^{-1/2}) U(t) - \frac{k}{2h}(T^{1/2} - T^{-1/2})] f(U(t))$$

Higher order methods have less aritficial diffusion!

$\partial_t u + b \partial_x u = 0$: Godunov: $\frac{C|b|}{2} h^2 \partial_x u$ Lax - Wendroff: $\frac{C^2 b^2}{2} h^2 \partial_x u$, $C = \frac{k}{h}$

Higher order methods II: Multi-step

$$\frac{du}{dt} = f(u,t), \quad U(t+k) = \sum_{s=0}^{S} \alpha_s U(t-sk) + k\beta_s f(U, t-sk)$$

Problem 1: needs start up!

Stability:

$$U(nk) = g^n U(0), \quad g^{n+1} = \sum_{s=0}^{S} \alpha_s g^{n-s} + k...$$

g solution of polynomial of degree $S+1$, $S+1$ roots!

Example: Leapfrog for $\partial_t u + \partial_x f(u)$

$$U(t+k) = U(t-k) - \frac{bk}{h}(T - T^{-1}) f(U(t)),$$

growth function g for $\partial_t f + b\partial_x f = 0$:

$$g^2 = 1 + 2igbC \sin(\xi h), \quad C = \frac{k}{h}$$

For $|b|C \leq 1$: $|g_{12}| = 1$, no artificial diffusion!!! (weakly unstable)

MONOTONICITY AND THE MAXIMUM PRINCIPLE

Difference scheme for general nonlinear problem

$$U_j(t+k) = g_j(U_1(t), .., U_N(t))$$

Consistency $\Rightarrow$
$g_j(u, ..., u) = u, \ \forall j$ (or $g_j(u, ..., u) = u(1 + O(k)), \ \forall j$ in the presence of source terms).

Lemma:

$$\frac{\partial g_j}{\partial u_k} \geq 0 \ \forall j, k \quad \Rightarrow \quad \|U(t+k)\|_\infty \leq \|U(t)\|_\infty,$$

Implicit schemes:

$$g_j^+(U(t+k)) = g_j^-(U(t))$$

$$g_j^+(.., U_s(t), ..) = g_j^+(.., U_s(t), ..), \ \forall j$$

Define:

$$g_j^+(v) = G_j^+(.., v, v, ..), \quad g_j^-(v) = G_j^-(.., v, v, ..)$$

Assume:

$$(A1) \quad \frac{\partial G_j^+}{\partial U_s} \leq 0 \text{ for } j \neq s, \quad \frac{\partial G_j^-}{\partial U_s} \geq 0 \ \forall j, s$$

$$(A2) \quad g_j^+(v) \geq g_j^-(w) \ \Rightarrow \ v \geq w \ \forall j$$

Lemma: Assumptions (A1),(A2) imply $\|U(t+k)\|_\infty \leq \|U(t)\|_\infty$

Monotone schemes build (nonlinear) averages over the solution with nonnegative coefficients!

Example: Lax - Wendroff

Methods of order higher than order 1 cannot be monotone!

THE TVD PROPERTY

The discrete W_1^∞ norm: $\int |\partial_x u| \, dx$

Non-oscillatory schemes:

Definition: A scheme is TVD $\iff$

$$\sum_j |(T-1)U_j^{n+1}| \le \sum_j |(T-1)U_j^n|$$

The linear case:
Write scheme solely in terms of derivatives.

$$U^{n+1} = U^n - T^{-1}(A^n V^n) + B^n V^n, \quad V^n := (T-1)U^n$$

Theorem:

$$A, B \ge 0, \quad A + B \le 1 \quad \Rightarrow \quad TVD$$

LECTURE 5

FLUX LIMITER METHODS

$$U^{n+1} = U^n - c(T-1)F^n$$

Idea:
Lower order method with numerical flux F^L. Higher order method with numerical flux F^H. Lower order method non-oscillatory.

Combine:

$$F = F^L + \Phi(F^H - F^L)$$

Smooth part: $\phi \approx 1$; Non-oscillatory otherwise, i.e. total method is TVD.

Total method formally only first order but higher order in smooth regions.

THE FLUX LIMITER

Condition 1: (Smooth order)

$$\Phi = \phi(\frac{(1 - T^{-1})U}{(T - 1)U}), \quad \phi(1) = 1$$

Condition 2: Choose ϕ such that total method is TVD.

Derive ϕ for linear case and use in general.

$$\partial_t u + a\partial_x u = 0, \quad a > 0$$

Example: Upwind and Lax Wendroff

$$F^L = aT^{-1}U, \quad F^H = \frac{a}{2}(1+T^{-1})U - \frac{ca^2}{2}(1-T^{-1})U$$

$$F^H - F^L = \frac{a}{2}(1-ac)(1-T^{-1})U$$

$$U^{n+1} = \{1 - \nu(1-T^{-1})[1 + \frac{\phi}{2}(1-\nu)(T-1)]\}U^n, \quad \nu := ac$$

$$U^{n+1} = U^n - \nu[1 - \frac{(1-\nu)}{2}T^{-1}\phi]T^{-1}V^n - \nu\frac{\phi}{2}(1-\nu)V^n, \quad V = (T-1)U$$

$$A = \nu[1 - \frac{(1-\nu)}{2}\phi] + \nu\frac{T\phi}{2}(1-\nu)\frac{TV}{V}$$

$$A = \nu\{1 + \frac{(1-\nu)}{2}(-\phi + T\phi\frac{TV}{V})\}$$

$$A \geq 0 \quad \Rightarrow \quad \left|\frac{\phi(\theta_j)}{\theta_j} - \phi(\theta_{j-1})\right| \leq 2, \quad \theta_j = \frac{T^{-1}V}{V}$$

Two standard choices:

1. Superbee:

$$\phi(\theta) = \max\{0, \min\{1, 2\theta\}, \min\{\theta, 2\}\}$$

2. Van Leer:

$$\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$$

LECTURE 6

BOUNDARY CONDITIONS AND GHOSTPOINTS FOR HYPERBOLIC SYSTEMS

$$\partial_t u + \partial_x [A(x,t)u] = 0, \quad x \geq 0, \quad Bu(0,t) = u_b,$$

$u \in \mathbb{R}^N$, $B \in \mathbb{R}^{K \times N}$

Relation of B to A: Influx given in terms of outflux !
Diagonalization:

$$A(0,t) = EDE^{-1}, \quad D = \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix}, \quad D_+ > 0, \; D_- \leq 0$$

Important!! $dim(D) = K \times K$
Projections: partition E

$$E = (E_+, E_-), \quad E_+ : N \times K, \quad E^{-1} = \begin{pmatrix} R_+ \\ R_- \end{pmatrix}, \quad R_+ : K \times N$$

Necessary: $\exists (BE_+)^{-1}$

Ghostpoint:

$$R_- U(-h,t) = R_-(2U(0,t) - U(h,t))$$

$$R_+ U(-h,t) = (BE_+)^{-1}[u_b - BE_- R_-(2U(0,t) - U(h,t))]$$

$$U(-h,t) = E \begin{pmatrix} (BE_+)^{-1} \\ 0 \end{pmatrix} u_b + E \begin{pmatrix} -(BE_+)^{-1} BE_- R_- \\ R_- \end{pmatrix} (2U(0,t) - U(h,t))$$