

Wigner transport models of the electron-phonon kinetics in quantum wires

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Two quantum-kinetic models of ultrafast electron transport in quantum wires are derived from the generalized electron-phonon Wigner equation. The various assumptions and approximations allowing to find closed equations for the reduced electron Wigner function are discussed with an emphasis on their physical relevance. The models correspond to the Levinson and the Barker-Ferry equations, now generalized to account for a space-dependent evolution. They are applied to study the quantum effects in the dynamics of an initial packet of highly non-equilibrium carriers, locally generated in the wire. The properties of the two model equations are compared and analyzed.

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I. INTRODUCTION

The early time dynamics of highly non-equilibrium confined carriers incorporates a variety of phenomena, which, reflecting the uncertainty relations, are beyond the Boltzmann-like picture of the transport process. An instantaneous scattering process occurring between electron states with well defined energy provides no more an adequate description: Effects of time dependent collisional broadening (CB) and retardation of phonon replicas have been investigated theoretically and experimentally in homogeneous semiconductors¹⁻⁸. These effects are related to the lack of energy conservation and the memory character of the electron-phonon dynamics, and are due to the finite duration of the interaction process. The effect of the action of the electric field during the process of collision - the intra-collisional field effect, (ICFE)⁹⁻¹², - has attracted the scientific attention quite some time. It has been shown that the intra-collisional field effect is not important in stationary high field transport in semiconductors when single-valley transport is considered,¹³. Rather, the effect must be sought in the time domain of the early time evolution, which precedes the formation of the classical energy conserving δ -function^{14,15}.

A natural representation in space homogeneous systems is provided by the wave vector space, which reflects the momentum conservation. Kinetic model equations accounting for these effects are developed in the framework of Green's function^{10,16-18} or density matrix formalisms^{1,19,20}. Explicitly or implicitly these approaches give rise the Levinson¹⁹ and/or the Barker-Ferry¹⁰ equations. Indeed, a general model for photo-excited semiconductors is the set of semiconductor Bloch equations for the electron, hole and phonon distributions and the inter-band polarization. Phenomena due to carrier-light, carrier-carrier and carrier-phonon interactions are accounted for in a comprehensive way⁸. The

case of a single-band model, at low density regime focuses of the carrier-phonon interaction. In this case the polarization can be eliminated giving rise to a generation rate, which furthermore is approximated by an initial condition. Obtained is a set of equations for the electron and phonon distributions and the phonon-assisted density matrix¹. Under the assumption of equilibrium phonons the set gives rise to the Barker-Ferry equation²¹. The later can be considered as a generalization of the Levinson equation, which accounts for the finite electron lifetime due to the interaction with the phonons.

Confined systems are characterized with small spatial scales where another basic assumption of classical transport - this for a scattering process occurring at a well defined position loses its validity. In such systems the scattering is no more local in space due to the finite duration of the carrier-phonon interaction. The introduction of a spatial coordinate allows to account for inhomogeneities which can be due to the structure of the sample or due to the confinement of the initial condition. In such cases a convenient description of the transport is given by the Wigner-function formalism. It retains most of the basic classical notions, in particular the concepts for phase space and a distribution function. Averaged values of the physical quantities in Boltzmann and Wigner pictures are expressed with the same functionals of the corresponding distribution functions. Another advantage of the Wigner picture is the opportunity to utilize Newton's trajectories along with the properties of the Liouville theorem. This allows to interpret the entire quantum evolution, and in particular the interaction with the Wigner potential, in terms of particles²². Both classical and quantum particles evolve over segments of Newton's trajectories. However since the nonlocal nature of the scattering, evolution trajectories inherent for Boltzmann particles are generalized by Wigner paths²³. The latter in principle lead to a different spatial distribution of the system. In general, in contrast to a classical distribution, the Wigner

function can have negative values which manifests the uncertainty relation and thus the quantum character of the evolution²⁴.

Physical systems which combine short spatial and time scales recently become a field of active research interest. An equation, which under homogeneous conditions reduces to the Levinson equation, has been derived within a Green's function approach by using the generalized Kadanoff-Baym Ansatz²⁵. Wigner equation models for nanometer and femtosecond transport regimes have been recently proposed in^{26, 27}. It has been shown that the case of a constant electric field gives rise to a Levinson type of equation which accounts for the spatial dimensions²⁶. The semiconductor Bloch equations have been generalized for spatially inhomogeneous excitations by using a density matrix approach¹⁵. The Wigner counterparts of the equations are then obtained by using a Fourier transform for both cases of two- and single-band models. The later gives rise to an inhomogeneous generalization of the Levinson equation. The models have been compared for the case of spatiotemporal evolution of a local electron distribution, optically generated in a quantum wire. The comparison shows that the typical quantum-kinetic features of carrier-phonon interaction in the single-band model and the physically comprehensive two band model are essentially the same. From here it can be concluded that the Levinson and Barker-Ferry models provide a relevant description of these features.

Quantum-kinetic effects in the evolution of carriers, confined in quantum wires have been investigated in the framework of the Barker-Ferry model derived with the help of a projection technique²⁸. Actually derived is the Levinson model and the factor accounting for the finite electron lifetime has been introduced by heuristic considerations²⁹. Memory effects have been neglected by taking the Markovian limit of the equation. This approach leads to a Boltzmann-like equation where the classical δ function is replaced by a Lorentzian. The equation has been solved by a modification of the classical Ensemble Monte Carlo method. The solutions, obtained at different evolution times well demonstrate stationary ICFE and CB effects. However, we note that at larger evolution times the approach based on stationary broadening of the energy conservation can lead to unphysical solutions. The reason is in the long reaching wings of the Lorentzian which can cause artificial heating of the carrier system³⁰.

In this contribution we utilize the Wigner formalism to derive the Levinson and the Barker-Ferry equations for the carrier-phonon kinetics in a quantum wire. First we obtain the generalized Wigner function (GWF)³¹ of the carrier-phonon system in the wire. Of interest is the reduced, or electron Wigner function obtained from the diagonal with respect to the phonon basis GWF elements. The latter are linked by the corresponding equation of motion to first off-diagonal (FOD) elements which differ by adding or subtracting a single phonon in given mode \mathbf{q} in the left or right basis. The FOD elements are

linked to second off-diagonal elements (SOD) etc., which gives rise to an infinite hierarchy of equations coupled by the phonon degrees of freedom. A closure at certain level is obtained by consecutive steps of assumptions and approximations. Frequently applied will be the random phase approximation which is used to neglect the rapidly oscillating in time terms. A Markov approximation will be used to derive the factor accounting for the finite carrier lifetime of the Barker-Ferry model. The trace operation is applied at the very end, after the truncation of the hierarchy, which is another peculiarity of our approach.

The physical relevance of the two models as well as the heuristic aspects of the formal steps of finding closed equations for the electron function are discussed in the second part of the paper. In particular the requirements ensuring an equilibrium phonon system are analyzed beyond the Bloch assumption.

Finally we present simulation results for the evolution of an initial electron distribution in a rectangular quantum wire. Extreme conditions of very low temperature are chosen. In this case the evolution of classical electrons is very transparent as they can only loose energy from the phonons. Moreover quantum electrons can be considered in the ground state in the plane normal to the wire. The classical evolution provides a background which is used to outline the quantum effects introduced by the explored models. Analyzed is the behavior of the physical observables corresponding to the first moments of the Wigner function: the concentration, the wave vector distribution and the energy density. The simulation results are obtained by a backward Monte Carlo method. The method allows a pointwise evaluation of the Wigner function and its moments with a desired precision.

II. THE GENERALIZED WIGNER EQUATION

A. Formulation of the transport problem

We consider a low density system of electrons evolving in a quantum wire and interacting with the lattice vibrations. The description of the system is provided by both electron and phonon degrees of freedom. We first generalize the Wigner function and the Wigner equation for the coupled electron-phonon system in the wire. The Hamiltonian of the system is given by

$$H = H_0 + V + H_p + H_{e-p} = -\frac{\hbar^2}{2m}\nabla_{\mathbf{r}} + V(\mathbf{r}) + \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \hbar \omega_{\mathbf{q}} + i\hbar \sum_{\mathbf{q}} F(\mathbf{q})(b_{\mathbf{q}} e^{i\mathbf{q}\hat{\mathbf{r}}} - b_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\hat{\mathbf{r}}}) \quad (1)$$

the free electron part H_0 , the wire potential $V(\mathbf{r})$ the free-phonon Hamiltonian H_p and the electron-phonon interaction H_{e-p} . Here $b_{\mathbf{q}}^\dagger$ and $b_{\mathbf{q}}$ are the creation and annihilation operators for the phonon mode \mathbf{q} , $\omega_{\mathbf{q}}$ is the energy of the mode, and $F(\mathbf{q})$ is the electron-phonon coupling element, which depends on the type of phonon

scattering analyzed. The state of the phonon subsystem is presented by the set $\{n_{\mathbf{q}}\} = \{n_{\mathbf{q}_1}, n_{\mathbf{q}_2}, \dots\}$ where $n_{\mathbf{q}}$ is the occupation number of the phonons in mode \mathbf{q} . Then the representation is given by the vectors $|\{n_{\mathbf{q}}\}, \mathbf{r}\rangle = |\{n_{\mathbf{q}}\}\rangle|\mathbf{r}\rangle$. A homogeneous electric field $E(t)$ can be applied along the z direction of the wire, the carriers are assumed confined in the normal plane. For a transparency of the presentation we assume a stationary electric field and a ground state Ψ in the normal plane.

$$H_0 + V(\mathbf{r}) = H_{\perp} + H_z = H_{0\perp} + V_{\perp} + H_{0z} + V(z);$$

where $H_{\perp}\Psi = E_{\perp}\Psi$, $V(z) = -eEz$ and $|\mathbf{r}\rangle = |\mathbf{r}_{\perp}\rangle|z\rangle$. The generalization for time dependent fields and a set of sub-bands is straightforward.

The electron-phonon Wigner function is defined by the Fourier transform of the density operator $\hat{\rho}_t$:

$$f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t) = \frac{1}{2\pi\hbar} \int dz' \int d\mathbf{r}_{\perp} e^{-ip_z z'/\hbar} \langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_{\perp} | \hat{\rho}_t | \mathbf{r}_{\perp} \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle$$

Since the separation of the transport task in the normal plane and the z direction we have $\hat{\rho}_t = |\Psi\rangle\langle\Psi| \hat{\rho}_{tz}$:

$$\langle \mathbf{r}, \{n_{\mathbf{q}}\} | \hat{\rho}_t | \{n'_{\mathbf{q}}\}, \mathbf{r}' \rangle = \Psi^*(\mathbf{r}'_{\perp}) \Psi(\mathbf{r}_{\perp}) \rho(z, z', \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)$$

Finally the requirement for normalization of Ψ leads to:

$$f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t) = \frac{1}{2\pi\hbar} \int dz' e^{-ip_z z'/\hbar} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)$$

B. Derivation of the generalized Wigner equation

The equation of motion of f_w is obtained from the von-Neumann equation for the density matrix:

$$\frac{\partial f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)}{\partial t} = \int \frac{dz'}{i2\pi\hbar^2} \int d\mathbf{r}_{\perp} e^{-ip_z z'/\hbar}$$

$$\begin{aligned} I &= \frac{1}{2\pi\hbar} \int d\mathbf{r}_{\perp} \int dz' \int d\mathbf{r}'' e^{-ip_z z'/\hbar} \langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_{\perp} | b_{\mathbf{q}'} e^{i\mathbf{q}'\mathbf{r}''} | \mathbf{r}'' \rangle \langle \mathbf{r}'' | \hat{\rho}_t | \mathbf{r}_{\perp} \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle \\ &= \sqrt{n_{\mathbf{q}'} + 1} \int d\mathbf{r}_{\perp} e^{i\mathbf{q}'_{\perp}\mathbf{r}_{\perp}} |\Psi(\mathbf{r}_{\perp})|^2 \int dz' e^{-ip_z z'/\hbar} e^{i\mathbf{q}'_{\perp}z(z+\frac{z'}{2})} \langle z + \frac{z'}{2}, \{n_1, \dots, n_{\mathbf{q}'} + 1, \dots\} | \hat{\rho}_{tz} | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle \\ &= \sqrt{n_{\mathbf{q}'} + 1} G(\mathbf{q}'_{\perp}) e^{i\mathbf{q}'_{\perp}z} f_w(z, p_z - \frac{\hbar\mathbf{q}'_{\perp}}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n'_{\mathbf{q}}\}, t) \end{aligned}$$

Here we used the normalization of the position basis $\langle \mathbf{r} | \mathbf{r}' \rangle = \delta(\mathbf{r} - \mathbf{r}')$ and the fact that $b_{\mathbf{q}}$ becomes a creation operator when operating to the left. The short notations $\{n_{\mathbf{q}}\}_{\mathbf{q}'}^{\pm} = \{n_1, \dots, n_{\mathbf{q}'} \pm 1, \dots\}$ are introduced. In this way

$$\langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_{\perp} | [H, \hat{\rho}_t]_{-} | \mathbf{r}_{\perp} \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle$$

For convenience we denote the right hand side of the above equation by $WT(H)$. In the following we evaluate $WT(H)$ for each term of the Hamiltonian (1). $WT(H_{\perp})$ gives zero since the assumed ground state in the normal direction. $WT(H_{0z})$ and $WT(-eEz)$ are readily calculated by using integration by parts:

$$WT(H_{0z}) = -\frac{p_z}{m} \frac{\partial f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)}{\partial z}$$

$$WT(-eEz) = -eE \frac{\partial f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)}{\partial p_z}$$

The free phonon term gives rise to:

$$WT(H_p) = \frac{1}{i\hbar} (\epsilon(\{n_{\mathbf{q}}\}) - \epsilon(\{n'_{\mathbf{q}}\})) f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t)$$

where $\epsilon(\{n_{\mathbf{q}}\}) = \sum_{\mathbf{q}} n_{\mathbf{q}} \hbar \omega_{\mathbf{q}}$. $WT(H_{e-p})$ consists of four contributions arising from the commutator of the density operator with the phonon creation and annihilation operators. They are evaluated with the help of the decomposition of the unity:

$$1 = \int dz'' |z'' \rangle \langle z''| \int d\mathbf{r}'_{\perp} |\mathbf{r}'_{\perp} \rangle \langle \mathbf{r}'_{\perp}|$$

As \mathbf{q} has been already used in the notation of the phonon basis, the phonon mode in H_{e-p} , (1) is replaced by \mathbf{q}' .

The first contribution is estimated as follows:

$\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+$ ($\{n_{\mathbf{q}}\}_{\mathbf{q}'}^-$) are states of the phonon subsystem, obtained from $\{n_{\mathbf{q}}\}$ by increasing (decreasing) the number

of phonons in the mode \mathbf{q}' by unity. In a similar way:

$$II = -\frac{1}{2\pi\hbar} \int d\mathbf{r}_\perp \int dz' e^{-ip_z z'/\hbar} \times \\ \langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_\perp | b_{\mathbf{q}'}^\dagger e^{-i\mathbf{q}'\mathbf{r}''} \hat{\rho}_t | \mathbf{r}_\perp \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle = \\ -\sqrt{n_{\mathbf{q}'}} G^*(\mathbf{q}'_\perp) e^{-iq'_z z} f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}^-, \{n'_{\mathbf{q}}\}, t)$$

We note that a half of the z component of the phonon mode is added to the electron momentum p_z . The sign of q'_z is opposite to the sign of the argument in the corresponding exponent. The phonon annihilation and creation operators of the next two contributions change the right phonon basis.

$$III = -\frac{1}{2\pi\hbar} \int d\mathbf{r}_\perp \int dz' e^{-ip_z z'/\hbar} \times \\ \langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_\perp | \hat{\rho}_t b_{\mathbf{q}'} e^{i\mathbf{q}'\mathbf{r}''} | \mathbf{r}_\perp \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle =$$

$$-\sqrt{n'_{\mathbf{q}'}} G(\mathbf{q}'_\perp) e^{iq'_z z} f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}^-, t)$$

Here the signs of the phonon mode added to p_z and in the exponent in front of f_w are the same. Finally the fourth term is:

$$IV = \frac{1}{2\pi\hbar} \int d\mathbf{r}_\perp \int dz' \int d\mathbf{r}'' e^{-ip_z z'/\hbar} \times \\ \langle z + \frac{z'}{2}, \{n_{\mathbf{q}}\} | \langle \mathbf{r}_\perp | \hat{\rho}_t b_{\mathbf{q}'}^\dagger e^{-i\mathbf{q}'\mathbf{r}''} | \mathbf{r}_\perp \rangle | \{n'_{\mathbf{q}}\}, z - \frac{z'}{2} \rangle = \\ \sqrt{n'_{\mathbf{q}'}} + 1 G^*(\mathbf{q}'_\perp) e^{-iq'_z z} f_w(z, p_z - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}^+, t)$$

By collecting all contributions we obtain the equation of motion for the GWF:

$$\left(\frac{\partial}{\partial t} + \frac{p_z}{m} \nabla_z + eE \nabla_{p_z} \right) f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t) = \frac{1}{i\hbar} (\epsilon(\{n_{\mathbf{q}}\}) - \epsilon(\{n'_{\mathbf{q}}\})) f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, t) + \sum_{\mathbf{q}'} F(\mathbf{q}') \times \\ \left\{ G(\mathbf{q}'_\perp) e^{iq'_z z} \sqrt{n_{\mathbf{q}'}} + 1 f_w(z, p_z - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}^+, \{n'_{\mathbf{q}}\}, t) - G^*(\mathbf{q}'_\perp) e^{-iq'_z z} \sqrt{n_{\mathbf{q}'}} f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}^-, \{n'_{\mathbf{q}}\}, t) \right. \\ \left. - G(\mathbf{q}'_\perp) e^{iq'_z z} \sqrt{n'_{\mathbf{q}'}} f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}^-, t) + G^*(\mathbf{q}'_\perp) e^{-iq'_z z} \sqrt{n'_{\mathbf{q}'}} + 1 f_w(z, p_z - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}^+, t) \right\} \quad (2)$$

The generalized Wigner equation couples an element $f_w(\dots, \{n\}, \{m\}, t)$ to four neighborhood elements with one phonon added or subtracted in the left or right basis for the particular mode \mathbf{q}' of the sum. For all modes \mathbf{q} the number of phonons $n_{\mathbf{q}}$ can be any integer between 0 and infinity and the sum over \mathbf{q}' couples all modes.

The equation can not be solved without relevant criteria which to neglect most of the GWF elements. A natural hierarchy in the set of elements is introduced by the fact that the pertinent physical information about the electron subsystem is provided by the main diagonal of (2): the reduced Wigner function is defined by the trace

$$f_w(z, p_z, t) = \sum_{\{n_{\mathbf{q}}\}} f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t) \quad (3)$$

of the generalized Wigner function $f_w(\dots, \{n\}, \{n\}, t)$, diagonal with respect to the phonon coordinates.

In this respect appropriate hypotheses are the assumptions for an initially decoupled electron-phonon system, weak interaction, and equilibrium phonons. A useful tool for neglecting terms is the random phase approximation (RPA)³² In a sum of the type $\sum_{\mathbf{q}, \mathbf{q}'} \exp\{i(f(\mathbf{q}) - f(\mathbf{q}'))t\}$ this approximation essentially retains the terms $\mathbf{q} = \mathbf{q}'$

and neglects the rapidly oscillating terms which more or less average to zero. Consider the imaginary term determined by the difference of the energies of the left and right phonon states. As seen from the integral form of the equation (Appendix A), this term is related to the frequency of the oscillations in time of the GWF. This frequency will be the main argument in the process of comparison of the consecutive terms in the hierarchy.

According to these considerations we must begin with the equation for the diagonal elements.

C. The Equation for the Diagonal Elements

By denoting $F'_G = F(\mathbf{q}')G(\mathbf{q}'_\perp)$ the equation reads:

$$\left(\frac{\partial}{\partial t} + \frac{p_z}{m} \nabla_z + eE \nabla_{p_z} \right) f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t) = \\ \sum_{\mathbf{q}'} \left\{ F'_G e^{iq'_z z} \sqrt{n_{\mathbf{q}'}} + 1 f_w(z, p_z - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}^+, \{n_{\mathbf{q}}\}, t) \right. \\ \left. - F'_G e^{-iq'_z z} \sqrt{n_{\mathbf{q}'}} f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}^-, \{n_{\mathbf{q}}\}, t) \right\} \quad (4)$$

$$\begin{aligned}
& -F_G' e^{iq_z z} \sqrt{n_{\mathbf{q}'}} f_w(z, p_z + \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^-, t) \\
& + F_G'^* e^{-iq_z z} \sqrt{n_{\mathbf{q}'} + 1} f_w(z, p_z - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, t) \}
\end{aligned}$$

The equation is accomplished by an initial condition, corresponding to a noninteracting system where the phonons are assumed in equilibrium.

We note the lack of the imaginary term, which means that diagonal elements do not show the oscillatory behavior typical for the off-diagonal elements. A diagonal element is linked to FOD elements, which are diagonal in all modes but the current mode \mathbf{q}' of the summation. In this mode the four neighbors of $n_{\mathbf{q}'}$, $n_{\mathbf{q}'}$ namely $n_{\mathbf{q}'} \pm 1$, $n_{\mathbf{q}'}$ and $n_{\mathbf{q}'}$, $n_{\mathbf{q}'} \pm 1$ are concerned. We need to consider only the FOD elements contained in the first two terms on the right: From the definition of f_w and (4) it follows that the third and the fourth terms are conjugated to

the second and first terms respectively.

In general the equations for $f_{FOD}^\pm = f_w(\cdot, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\})$ introduce SOD elements. We pursue the idea to truncate the hierarchy by neglecting the second and higher order off-diagonal elements. It is necessary to write explicitly all algebraic details in order to recognize the few terms which must remain in the right hand sides of the equations.

III. TRUNCATION AT THE FOD LEVEL

A. The Equations for the FOD Elements

In what follows we use the abbreviation $F_G'' = F_G(\mathbf{q}'')$. The equations of motion for f_{FOD}^+ and f_{FOD}^- can be unified as follows:

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{p_z \mp \frac{\hbar q_z'}{2}}{m} \nabla_z + eE \nabla_{p_z} \pm i\omega_{\mathbf{q}'} \right) f_w(z, p_z - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\}, t) = \sum_{\mathbf{q}''} \left\{ F_G'' e^{iq_z'' z} \sqrt{n_{\mathbf{q}''} + 1} \right. \\
& f_w(z, p_z \mp \frac{\hbar(q_z' \pm q_z'')}{2}, \{ \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm \}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}, t) - F_G''^* e^{-iq_z'' z} \sqrt{n_{\mathbf{q}''}} f_w(z, p_z \mp \frac{\hbar(q_z' \mp q_z'')}{2}, \{ \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm \}_{\mathbf{q}''}^-, \{n_{\mathbf{q}}\}, t) - F_G'' e^{iq_z'' z} \\
& \left. \sqrt{n_{\mathbf{q}''}} f_w(z, p_z \mp \frac{\hbar(q_z' \mp q_z'')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\}_{\mathbf{q}''}^-, t) + F_G''^* e^{-iq_z'' z} \sqrt{n_{\mathbf{q}''} + 1} f_w(z, p_z \mp \frac{\hbar(q_z' \pm q_z'')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\}_{\mathbf{q}''}^+, t) \right\}
\end{aligned} \quad (5)$$

Accordingly we refer to the separate equations as (5 $^\pm$). On this stage we approximate the equations by neglecting all SOD elements on the right. The only elements in (5 $^+$), which are not SOD are obtained from the second and the fourth terms on the right in the case $\mathbf{q}'' = \mathbf{q}'$. In particular the factor $\sqrt{n_{\mathbf{q}''}}$ becomes $\sqrt{n_{\mathbf{q}'} + 1}$ as by default the number of phonons in \mathbf{q}' mode is raised by unity, Similarly, in (5 $^-$) the first and the third term on the right recover diagonal elements if $\mathbf{q}'' = \mathbf{q}'$. The factor $\sqrt{n_{\mathbf{q}''} + 1}$ becomes $\sqrt{n_{\mathbf{q}'}}$ since the number of phonons in \mathbf{q}' mode is lowered by unity.

We will make use of the integral forms of the approximated equations. According Appendix A the integral formulation is obtained with the help of the characteristics of the Liouville operators in (5). These are Newton's trajectories, (43), initialized by the time and phase space variables which identify the left hand sides of (5 $^\pm$). Thus the particular trajectories are initialized by $(z, p_z \mp \frac{\hbar q_z'}{2}, t)$:

$$\begin{aligned}
z^\mp(t') &= z - \frac{1}{m} \int_{t'}^t p_z^\mp(\tau) d\tau \\
p_z^\mp(t') &= p_z \mp \frac{\hbar q_z'}{2} - eE(t - t') = p_z(t') \mp \frac{\hbar q_z'}{2}
\end{aligned} \quad (6)$$

As the electron-phonon system is initially decoupled, the initial conditions for f_{FOD}^\pm are zero:

$$\begin{aligned}
& f_w(z, p_z \mp \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\}, t) = \\
& F_G^\mp(\mathbf{q}') \sqrt{n_{\mathbf{q}'} + \frac{1}{2} \pm \frac{1}{2}} \int_0^t dt' e^{\mp i\omega_{\mathbf{q}'}(t-t')} e^{\mp iq_z' z^\mp(t')} \\
& \left\{ \mp f_w(z^\mp(t'), p_z(t'), \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t') \right. \\
& \left. \pm f_w(z^\mp(t'), p_z(t') \mp \hbar q_z', \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^\pm, t') \right\}
\end{aligned} \quad (7)$$

Here $F_G^+ = F_G$, $F_G^- = F_G^*$ and consistently with (6) $p_z(t') = p_z - eE(t - t')$.

B. The Inhomogeneous Levinson Equation

A replacement of (7) into (4) yields to an equation, which contains only diagonal elements of the GWF:

$$\left(\frac{\partial}{\partial t} + \frac{p_z}{m}\nabla_z + eE\nabla_{p_z}\right) f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t) = 2Re \left(\sum_{\mathbf{q}'} |F_G(\mathbf{q}')|^2 \int_0^t dt' \left\{ \right. \right. \quad (8)$$

$$\left. \left. (n_{\mathbf{q}'} + 1)e^{iq'_z z} e^{-i\omega_{\mathbf{q}'}(t-t')} e^{-iq'_z z^-(t')} \left(f_w(z^-(t'), (p_z - \hbar q'_z)(t'), \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+ \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, t') - f_w(z^-(t'), p_z(t'), \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t') \right) \right. \right.$$

$$\left. \left. - n_{\mathbf{q}'} e^{-iq'_z z} e^{i\omega_{\mathbf{q}'}(t-t')} e^{iq'_z z^+(t')} \left(f_w(z^+(t'), p_z(t'), \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t') - f_w(z^+(t'), p_z(t') + \hbar q'_z, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^- \{n_{\mathbf{q}}\}_{\mathbf{q}'}^-, t') \right) \right\} \right)$$

The arguments of the exponent are evaluated with the help of (6):

$$\begin{aligned} & \pm q'_z z \mp \omega_{\mathbf{q}'}(t-t') \mp q'_z z^\mp(t') = \\ & - \int_{t'}^t \left(\frac{q'_z}{m} \left(p_z - eE(t-\tau) \mp \frac{\hbar q'_z}{2} \right) \mp \omega_{\mathbf{q}'} \right) d\tau = \\ & \mp \int_{t'}^t \frac{1}{\hbar} (\epsilon(p_z(\tau)) - \epsilon(p_z(\tau) \mp \hbar q'_z) \mp \hbar \omega_{\mathbf{q}'}) d\tau \end{aligned}$$

Furthermore we modify (8) by switching the sign of \mathbf{q}' in the last row (for this we rely on the symmetry of $\omega_{\mathbf{q}'}$ and F_G) and by introducing the variable $p'_z = p_z - \hbar q'_z$.

The electron Wigner function (3) is obtained with the help of the assumption that the phonon system remains in equilibrium during the evolution. Formally this means that the variables in the diagonal elements can be separated as follows:

$$f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t') = f_w(z, p_z, t') \prod_{\mathbf{q}} P_{eq}(n_{\mathbf{q}}); \quad (9)$$

Here $P_{eq}(n_{\mathbf{q}})$ is the equilibrium probability for finding $n_{\mathbf{q}}$ phonons in mode \mathbf{q} . The mean equilibrium phonon number $n(\mathbf{q})$ is given by the Bose-Einstein distribution:

$$n(\mathbf{q}) = \sum_{n_{\mathbf{q}}=0}^{\infty} n_{\mathbf{q}} P_{eq}(n_{\mathbf{q}}); \quad \sum_{n_{\mathbf{q}}=0}^{\infty} P_{eq}(n_{\mathbf{q}}) = 1 \quad (10)$$

We replace (9) into the modified equation (8) and perform the trace operation. The phonon coordinates are encountered with the help of the equalities:

$$n(\mathbf{q}) + \frac{1}{2} \mp \frac{1}{2} = \sum_{n_{\mathbf{q}}} (n_{\mathbf{q}} + \frac{1}{2} \pm \frac{1}{2}) P_{eq}(n_{\mathbf{q}} \pm 1) \quad (11)$$

Obtained is an equation of Levinson type:

$$\left(\frac{\partial}{\partial t} + \frac{p_z}{m}\nabla_z + eE\nabla_{p_z}\right) f_w(z, p_z, t) = \quad (12)$$

$$\sum_{\mathbf{q}'_{\perp}, p'_z} \int_0^t dt' \left\{ S(p_z, p'_z, \mathbf{q}'_{\perp}, t, t') f_w(z^-(t'), p_z'(t'), t') \right.$$

$$\left. - S(p'_z, p_z, \mathbf{q}'_{\perp}, t, t') f_w(z^-(t'), p_z(t'), t') \right\}$$

$$S(p_z, p'_z, \mathbf{q}'_{\perp}, t, t') = 2|F_G(\mathbf{q}')|^2 \times \quad (13)$$

$$\left[n(\mathbf{q}) \cos \left(\int_{t'}^t \frac{(\epsilon(p_z(\tau)) - \epsilon(p'_z(\tau)) - \hbar \omega_{\mathbf{q}'}) d\tau}{\hbar} \right) + \right.$$

$$\left. (n(\mathbf{q}) + 1) \cos \left(\int_{t'}^t \frac{(\epsilon(p_z(\tau)) - \epsilon(p'_z(\tau)) + \hbar \omega_{\mathbf{q}'}) d\tau}{\hbar} \right) \right]$$

which describes the inhomogeneous evolution of carriers excited in a quantum wire. The discussion of its properties is postponed for section V.

The physically transparent way of the derivation of (12) encourages to make a further step and to take into account for the next level in the hierarchy of linked GWF elements.

IV. TRUNCATION AT THE SOD LEVEL

A. Closure of the equation for f_{FOD}^+

We first consider (5^+) and look for a criteria which to retain also certain SOD elements on the right hand side of the equation. Four types of such elements, shortly denoted by f_{SOD}^{++} , f_{SOD}^{+-} , $f_{SOD}^{+,-}$ and f_{SOD}^{++} , appear consecutively in the terms on the right. In this notations the comma separates the left from the right basis so that $++$, means two extra phonons in states \mathbf{q}' and \mathbf{q}'' in the left basis etc.. f_{SOD}^{+-} and $f_{SOD}^{+,-}$, which give rise to the only diagonal elements, have already been used above. We then analyze the SOD elements of the first and the third terms: f_{SOD}^{++} and $f_{SOD}^{+,-}$. They contain two extra phonons in the left basis. The corresponding equations of motion involve a frequency of $i2\omega$ added to the Liouville operator on the left hand side. As compared to the diagonal elements, these SOD elements oscillate rapidly in time. A straightforward application of RPA requires they to be neglected. Then the second and fourth terms remain to be squeezed to give rise of additional to the diagonal elements corrections. However a careful analysis shows that also f_{SOD}^{++} and $f_{SOD}^{+,-}$ give contributions: it is correct to compare elements linked within an equation. Jumps in the hierarchy as e.g. a comparison between SOD and diagonal elements can lead to erroneous conclusions.

We evaluate consecutively the contributions from the SOD elements to the right hand side of equation (5⁺). The corresponding equations of motion in general introduce third off-diagonal elements. We follow the same strategy for approximation of the right hand sides of these equations. Namely we consider only special cases, where third off-diagonal elements reduce to FOD elements. Then the SOD equations are solved and replaced in (5⁺). Obtained is an equation which contains FOD

elements only. This equation along with the counterpart obtained from (5⁻) are used to close (4). In what follows we need to assume a constant phonon frequency $\omega_{\mathbf{q}} = \omega$.

1. The contribution from f_{SOD}^{++} .

We begin with the equation of motion of f_{SOD}^{++} .

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \frac{p_z - \frac{\hbar(q'_z + q''_z)}{2}}{m} \nabla_z + eE \nabla_{p_z} + i2\omega \right) f_w(z, p_z - \frac{\hbar(q'_z + q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}, t) = \\ & + \sum_{\mathbf{q}'''} \left\{ F_G(\mathbf{q}''') e^{iq''_z z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q'_z + q''_z + q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}, t) \right. \\ & - F_G^*(\mathbf{q}''') e^{-iq''_z z} \sqrt{n_{\mathbf{q}'''} } f_w(z, p_z - \frac{\hbar(q'_z + q''_z - q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^-, \{n_{\mathbf{q}}\}, t) \\ & - F_G(\mathbf{q}''') e^{iq''_z z} \sqrt{n_{\mathbf{q}'''} } f_w(z, p_z - \frac{\hbar(q'_z + q''_z - q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}^-, t) \\ & \left. + F_G^*(\mathbf{q}''') e^{-iq''_z z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q'_z + q''_z + q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}^+, t) \right\} \end{aligned} \quad (14)$$

We approximate this equation by neglecting all terms but these containing FOD elements. Only two terms on the right can give rise to FOD elements. These are the second and the fourth term, provided that $\mathbf{q}''' = \mathbf{q}''$ or $\mathbf{q}''' = \mathbf{q}'$. The equation is integrated with the help of a

trajectory obtained from (6) for the initialization point $(z, p_z - \frac{\hbar(q'_z + q''_z)}{2}, t)$ identifying the Liouville operator in (14). The free term is zero, as the electron-phonon system is initially decoupled:

$$\begin{aligned} & F_G(\mathbf{q}'') e^{iq''_z z} f_w(z, p_z - \frac{\hbar(q'_z + q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}, t) = \\ & - \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{iq''_z z} e^{-iq''_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t') \\ & - \int_0^t dt' F_G^*(\mathbf{q}') F_G(\mathbf{q}'') e^{iq''_z z} e^{-iq'_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q''_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t') \\ & + \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{iq''_z z} e^{-iq''_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''} } f_w(z(t'), p_z(t') - \frac{\hbar(q'_z + 2q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}''}^+, t') \\ & + \int_0^t dt' F_G^*(\mathbf{q}') F_G(\mathbf{q}'') e^{iq''_z z} e^{-iq'_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}'} } f_w(z(t'), p_z(t') - \frac{\hbar(2q'_z + q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, t') \end{aligned} \quad (15)$$

(15) can be replaced in (5⁺), so that its right hand side appears as a correction attached to the right hand side of (5⁺). We evaluate the oscillations in time of the correction terms. However the integro-differential form is not convenient for an analysis in the time domain. In particular the left hand side of (5⁺) depends on $i\omega$ while the

correction terms depend on $e^{-i2\omega(t-t')}$. Convenient for this purpose is the integral form of the equation, which appears to be (7⁺) with additional terms on the right hand side arising from (15). The corrections due to these terms are additive, so that their contributions can be evaluated separately. We begin with the first term in (15) rewritten so that the q''_z dependent arguments of the

exponents are expressed in terms of electron energies:

$$q_z'' z - q_z'' z(t') = \int_{t'}^t \frac{\epsilon(p_z(\tau) - \frac{\hbar q_z'}{2}) - \epsilon(p_z(\tau) - \frac{\hbar q_z'}{2} - \hbar q_z'')}{\hbar} d\tau \quad (16)$$

For convenience of the notations we denote the energy difference in the numerator by $\Delta\epsilon(\tau)$. The term under considerations gives rise to the following contribution to the right hand side of (7⁺):

$$f_w(z, p_z - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t) = \dots - \sum_{\mathbf{q}''} (n_{\mathbf{q}''} + 1) |F_G(\mathbf{q}'')|^2 \int_0^t dt' \int_0^{t'} dt'' e^{-i\omega(t-t')} e^{-i2\omega(t'-t'')} \times e^i \int_{t''}^{t'} \frac{\Delta\epsilon(\tau)d\tau}{\hbar} f_w(z(t', t''), p_z(t'') - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t'') \quad (17)$$

where $z(t', t'')$ is expressed with the help of (6):

$$z(t', t'') = z^-(t') - \frac{1}{m} \int_{t''}^{t'} (p_z(\tau) - \frac{\hbar(q_z' + q_z'')}{2}) d\tau \quad (18)$$

(17) has the convenient form for analysis of the time dependences. The time integrals can be transformed with the help of the equality $\int_0^t dt' \int_0^{t'} dt'' = \int_0^t dt'' \int_{t''}^t dt'$:

$$\int_0^t dt' \int_0^{t'} dt'' e^{-i\omega(t-t')} e^{-i2\omega(t'-t'')} e^i \int_{t''}^{t'} \frac{\Delta\epsilon(\tau)d\tau}{\hbar} f_w = \int_0^t dt'' e^{-i\omega(t-t'')} \int_{t''}^t dt' e^i \int_{t''}^{t'} \frac{(\Delta\epsilon(\tau) - \hbar\omega)d\tau}{\hbar} f_w \quad (19)$$

Next the inner integral is approximated by taking the classical limit $\hbar \rightarrow 0$. The limit is discussed in details in Appendix B. It gives rise to a δ function and a principle value according to the formal relation:

$$\lim_{\hbar \rightarrow 0} \frac{1}{\hbar} \int_0^t d\tau e^{i\frac{\epsilon\tau}{\hbar}} = \pi\delta(\epsilon) + v.p. \frac{i}{\epsilon},$$

We note that this limit conveniently sets t' to t'' so that $z(t', t'')$ in f_w , (17), becomes $z^-(t'')$. If only the delta function is accounted for, (17) takes the form:

$$f_w(z, p_z - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t) = \dots - \int_0^t dt'' e^{-i\omega(t-t'')} \sum_{\mathbf{q}''} (n_{\mathbf{q}''} + 1) |F_G(\mathbf{q}'')|^2 \pi \hbar \times$$

$$F_G(\mathbf{q}'') e^{iq_z'' z} f_w(z, p_z - \frac{\hbar(q_z' - q_z'')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}_{\mathbf{q}''}, t) = - \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{iq_z'' z} e^{-iq_z'' z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''}} f_w(z(t'), p_z(t') - \frac{\hbar(q_z' - 2q_z'')}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}_{\mathbf{q}''}, t')$$

$$\delta \left(\epsilon \left(p_z(t'') - \frac{\hbar q_z'}{2} \right) - \epsilon \left(p_z(t'') - \frac{\hbar q_z'}{2} - \hbar q_z'' \right) - \hbar\omega \right) f_w(z(t''), p_z(t'') - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t'')$$

The obtained equation is of the type of (45), which along with equation (47) have a common differential counterpart given by (46), Appendix A. In other words if we take the time derivative of (20), we obtain (5⁺) with an additional term $\gamma_e f_{FOD}^+$ appearing in the right hand side of the equation. If transferred to the left this term gives rise to the correction:

$$\gamma_e (p_z - \frac{\hbar q_z'}{2}) = \sum_{\mathbf{q}''} (n_{\mathbf{q}''} + 1) |F_G(\mathbf{q}'')|^2 \quad (21)$$

$$\pi \hbar \delta \left(\epsilon \left(p_z - \frac{\hbar q_z'}{2} \right) - \epsilon \left(p_z - \frac{\hbar q_z'}{2} - \hbar q_z'' \right) - \hbar\omega \right)$$

added to the Liouville operator in the bracket of (5⁺). We note that the summation over \mathbf{q}'' involves only positive contributions to γ_e . The imaginary term with the principal value affects the phonon frequency - an effect known as a polaron shift in the energies. Here this effect is neglected.

Consider the rest of the terms in (15). The pre-factor of the third term in (15) can be evaluated in the same way leading to (20). However now the corresponding GWF element depends on \mathbf{q}_z'' and thus would contribute as a complex quantity to the sum in (5⁺). The same holds for the GWF elements in the second and the fourth terms, which moreover depend on the oscillatory pre-factor $e^{i(q_z'' - q_z')z}$. By referring to the RPA we can neglect these three terms.

Essentially the same steps will be applied for the rest of the terms. Next we evaluate the contribution from the third term since the derivations closely follow the already considered steps.

2. The contribution from $f_{SOD}^{+,-}$

We write down the equation for $f_w(z, p_z - \frac{\hbar(q_z' - q_z'')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}_{\mathbf{q}''}, t)$ and consider only the FOD elements which appear on the right. The equation is integrated with the help of a trajectory initialized by the phase space variables $(z, p_z - \frac{\hbar(q_z' - q_z'')}{2}, t)$. The solution is obtained explicitly in terms of FOD elements:

$$\begin{aligned}
& - \int_0^t dt' F_G^*(\mathbf{q}') F_G(\mathbf{q}'') e^{iq'_z z} e^{-iq'_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''} + 1} f_w(z(t'), p_z(t')) + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, t') \\
& + \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{iq'_z z} e^{-iq'_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''}} f_w(z(t'), p_z(t')) - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t') \\
& + \int_0^t dt' F_G^*(\mathbf{q}') F_G(\mathbf{q}'') e^{iq'_z z} e^{-iq'_z z(t')} e^{-i2\omega(t-t')} \sqrt{n_{\mathbf{q}''} + 1} f_w(z(t'), p_z(t')) - \frac{\hbar(2q'_z - q''_z)}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}''}\}_{\mathbf{q}'}, t')
\end{aligned}$$

Only the third term on the right survives after the RPA. Taking into account the particular trajectory $z(t')$ we can express $q''_z z - q'_z z(t')$ in the exponent as

$$- \int_{t'}^t \frac{\epsilon(p_z(\tau) - \frac{\hbar q'_z}{2}) - \epsilon(p_z(\tau) - \frac{\hbar q'_z}{2} + \hbar q''_z)}{\hbar} d\tau \quad (22)$$

This differs from (16) by the front sign and the sign of q''_z . As before we denote the energy difference by $\Delta\epsilon(\tau)$ and evaluate the contribution of the considered term to the integral form of (5⁺). Obtained is an expression, which differs from (17) in the following: (i) the factor $(n_{\mathbf{q}''} + 1)$ is replaced by $n_{\mathbf{q}''}$; (ii) $\Delta\epsilon(\tau)$ is updated according to (22); (iii) $z(t', t'')$ is updated from (18) by changing the sign of q''_z from plus to minus. The evaluation of the time integrals follows exactly (19). By taking the classical limit and following the arguments leading to (21), we obtain that the term

$$\gamma_a(p_z - \frac{\hbar q'_z}{2}) = \sum_{\mathbf{q}''} n_{\mathbf{q}''} |F_G(\mathbf{q}'')|^2 \pi \hbar \quad (23)$$

$$\delta \left(\epsilon \left(p_z - \frac{\hbar q'_z}{2} \right) - \epsilon \left(p_z - \frac{\hbar q'_z}{2} + \hbar q''_z \right) + \hbar \omega \right)$$

must be added to the Liouville operator in (5⁺). The phonon energy is now added to $\Delta\epsilon$ due to the minus sign in front of (22),

3. The correction from f_{SOD}^{+-}

The second term in (5⁺) already provided a diagonal element. We pursue what additional contribution δf_{SOD}^{+-} can be obtained from the corresponding equation of motion. Apparently δf_{SOD}^{+-} is

$$(1 - \delta_{\mathbf{q}', \mathbf{q}''}) f_w(z, p_z - \frac{\hbar(q'_z - q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}, t)$$

so that the equation of motion assumes that $\mathbf{q}' \neq \mathbf{q}''$:

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{p_z - \frac{\hbar(q'_z - q''_z)}{2}}{m} \nabla_z + eE \nabla_{p_z} \right) \delta f_{SOD}^{+-}(z, p_z - \frac{\hbar(q'_z - q''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}, t) = \quad (24) \\
& + \sum_{\mathbf{q}'''} \left\{ F_G(\mathbf{q}''') e^{iq'''_z z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q'_z - q''_z + q'''_z)}{2}, \{\{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}\}_{\mathbf{q}'''}, \{n_{\mathbf{q}}\}, t) \right. \\
& - F_G^*(\mathbf{q}''') e^{-iq'''_z z} \sqrt{n_{\mathbf{q}'''}} f_w(z, p_z - \frac{\hbar(q'_z - q''_z - q'''_z)}{2}, \{\{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}\}_{\mathbf{q}'''}, \{n_{\mathbf{q}}\}, t) \\
& - F_G(\mathbf{q}''') e^{iq'''_z z} \sqrt{n_{\mathbf{q}'''}} f_w(z, p_z - \frac{\hbar(q'_z - q''_z - q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}, t) \\
& \left. + F_G^*(\mathbf{q}''') e^{-iq'''_z z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q'_z - q''_z + q'''_z)}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}, t) \right\}
\end{aligned}$$

The four terms on the right hand side contain FOD elements obtained respectively by the combinations $\mathbf{q}''' = \mathbf{q}''$, $\mathbf{q}''' = \mathbf{q}'$, $\mathbf{q}''' = \mathbf{q}''$, $\mathbf{q}''' = \mathbf{q}'$. We recall that in this case $n_{\mathbf{q}'''}$ must be updated according to the actual number of phonons in modes \mathbf{q}' or \mathbf{q}'' respectively. The

reduced equation is integrated with the help of the trajectory $(z(t'), p_z(t'))$ initialized by the phase space variables $(z, p_z - \frac{\hbar(q'_z - q''_z)}{2}, t)$. The correction δf_{SOD}^{+-} is expressed in terms of FOD elements:

$$\begin{aligned}
& F_G^*(\mathbf{q}'') e^{-iq_z'' z} \delta f_{SOD}^+(z, p_z - \frac{\hbar(q_z' - q_z'')}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}, \{n_{\mathbf{q}}\}, t) = \\
& + \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{-iq_z'' z} e^{iq_z' z(t')} \sqrt{n_{\mathbf{q}''}} f_w(z(t'), p_z(t') - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t') \\
& - \int_0^t dt' F_G^*(\mathbf{q}') F_G^*(\mathbf{q}'') e^{-iq_z'' z} e^{-iq_z' z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') + \frac{\hbar q_z''}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}''}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t') \\
& - \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{-iq_z'' z} e^{iq_z' z(t')} \sqrt{n_{\mathbf{q}''}} f_w(z(t'), p_z(t') + \hbar q_z'' - \frac{\hbar q_z'}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}''}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, t') \\
& + \int_0^t dt' F_G^*(\mathbf{q}'') F_G^*(\mathbf{q}') e^{-iq_z'' z(t')} e^{-iq_z' z} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \hbar q_z' + \frac{\hbar q_z''}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, t')
\end{aligned} \tag{25}$$

We evaluate the contribution of the first term to the equation (7). As before the arguments of the exponents introduce the energy difference $\Delta\epsilon(\tau)$

$$q_z'' z(t') - q_z'' z = \int_{t'}^t \frac{\epsilon(p_z(\tau) - \frac{\hbar q_z'}{2}) - \epsilon(p_z(\tau) - \frac{\hbar q_z'}{2} + \hbar q_z'')}{\hbar} d\tau \tag{26}$$

under the time integral. The pursued contribution is obtained as follows:

$$\begin{aligned}
& f_w(z, p_z - \frac{\hbar q_z'}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t) = \dots - \sum_{\mathbf{q}''} n_{\mathbf{q}''} |F_G''|^2 \\
& \int_0^t dt' \int_0^{t'} dt'' e^{-i\omega(t-t')} e^{i \int_{t''}^{t'} \frac{\Delta\epsilon(\tau) d\tau}{\hbar}} \\
& f_w(z(t', t''), p_z(t''), \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t'')
\end{aligned} \tag{27}$$

where

$$z(t', t'') = z^-(t') - \frac{1}{m} \int_{t''}^{t'} (p_z(\tau) - \frac{\hbar(q_z' - q_z'')}{2}) d\tau \tag{28}$$

The time integrals are processed accordingly:

$$\int_0^t dt' \int_0^{t'} dt'' e^{-i\omega(t-t')} e^{i \int_{t''}^{t'} \frac{\Delta\epsilon(\tau) d\tau}{\hbar}} f_w = \tag{29}$$

$$\int_0^t dt'' e^{-i\omega(t-t'')} \int_{t''}^t dt' e^{i \int_{t''}^{t'} \frac{(\Delta\epsilon(\tau) + \hbar\omega) d\tau}{\hbar}} f_w$$

After taking the classical limit in the inner integral and neglecting the principal value we obtain the contribution to the Liouville operator in (5⁺):

$$\begin{aligned}
& \gamma_a(p_z - \frac{\hbar q_z'}{2}) = \sum_{\mathbf{q}''} n_{\mathbf{q}''} |F_G(\mathbf{q}'')|^2 \pi \hbar \\
& \delta \left(\epsilon \left(p_z - \frac{\hbar q_z'}{2} \right) - \epsilon \left(p_z - \frac{\hbar q_z'}{2} + \hbar q_z'' \right) + \hbar\omega \right)
\end{aligned}$$

The rest of the terms in (25) are neglected with the help of the RPA.

4. The correction from $f_{SOD}^{+,+}$

The correction $\delta f_{SOD}^{+,+}$ originated from forth term in (7) satisfies the equation

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{p_z - \frac{\hbar(q_z' + q_z'')}{2}}{m} \nabla_z + eE \nabla_{p_z} \right) \delta f_{SOD}^{+,+}(z, p_z - \frac{\hbar(q_z' + q_z'')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}''}^+, t) = \\
& + \sum_{\mathbf{q}'''} \left\{ F_G(\mathbf{q}''') e^{iq_z''' z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q_z' + q_z'' + q_z''')}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}'''}, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}^+, t) \right. \\
& - F_G^*(\mathbf{q}''') e^{-iq_z''' z} \sqrt{n_{\mathbf{q}'''}} f_w(z, p_z - \frac{\hbar(q_z' + q_z'' - q_z''')}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}'''}, \{n_{\mathbf{q}}\}_{\mathbf{q}'''}^+, t) \\
& - F_G(\mathbf{q}''') e^{iq_z''' z} \sqrt{n_{\mathbf{q}'''}} f_w(z, p_z - \frac{\hbar(q_z' + q_z'' - q_z''')}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'''}^+\}_{\mathbf{q}'''}, t) \\
& \left. + F_G^*(\mathbf{q}''') e^{-iq_z''' z} \sqrt{n_{\mathbf{q}'''} + 1} f_w(z, p_z - \frac{\hbar(q_z' + q_z'' + q_z''')}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'''}^+\}_{\mathbf{q}'''}, t) \right\}
\end{aligned} \tag{30}$$

under the condition $\mathbf{q}' \neq \mathbf{q}''$. The four terms on the right hand side contain FOD elements obtained respectively by the combinations $\mathbf{q}''' = \mathbf{q}'$, $\mathbf{q}''' = \mathbf{q}'$, $\mathbf{q}''' = \mathbf{q}''$, $\mathbf{q}''' = \mathbf{q}'$.

The approximated equation is integrated with the help of a trajectory initialized by $(z, p_z - \frac{\hbar(q'_z + q''_z)}{2}, t)$:

$$\begin{aligned}
& F_G^*(\mathbf{q}'') e^{-iq''_z z} \delta f_{SOD}^{+,+}(z, p_z - \frac{\hbar(q'_z + q''_z)}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}''}^+, t) = \\
& + \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{-iq''_z z} e^{iq''_z z(t')} \sqrt{n_{\mathbf{q}''} + 1} f_w(z(t'), p_z(t') - \hbar q''_z - \frac{\hbar q'_z}{2}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}''}^+\}_{\mathbf{q}'}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}''}^+\}_{\mathbf{q}'}, t') \\
& - \int_0^t dt' F_G^*(\mathbf{q}') F_G^*(\mathbf{q}'') e^{-iq''_z z} e^{-iq'_z z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q''_z}{2}, \{n_{\mathbf{q}}\}, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}, t') \\
& - \int_0^t dt' |F_G(\mathbf{q}'')|^2 e^{-iq''_z z} e^{iq''_z z(t')} \sqrt{n_{\mathbf{q}''} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t') \\
& + \int_0^t dt' F_G^*(\mathbf{q}'') F_G^*(\mathbf{q}') e^{-iq'_z z(t')} e^{-iq''_z z} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \hbar q'_z - \frac{\hbar q''_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{\{n_{\mathbf{q}}\}_{\mathbf{q}'}^+\}_{\mathbf{q}''}, t')
\end{aligned} \tag{31}$$

An application of the RPA filters all terms but the third. Taking into account the particular form of the trajectory $z(t')$ we obtain the exponential factor $-q''_z z + q'_z z(t')$ expressed in terms of electron energies:

$$- \int_{t'}^t \frac{\left(\epsilon(p_z(\tau) - \frac{\hbar q'_z}{2}) - \epsilon(p_z(\tau) - \frac{\hbar q''_z}{2} - \hbar q'_z) \right) d\tau}{\hbar} \tag{32}$$

This expression differs from (26) by the front sign and the sign of q''_z . The contribution of $\delta f_{SOD}^{+,+}$ to the FOD element accounted in the integral form of equation (5⁺) gives rise to an expression which differs from (27) in the following: (i) the factor $n_{\mathbf{q}''}$ in front of $|F_G|^2$ becomes $(n_{\mathbf{q}''} + 1)$; (ii) the energy difference in the exponent is

updated according to (32) (iii) $z(t', t'')$ is updated according to (28) by changing the sign of q''_z from plus to minus. Processing the time integrals as in equation (29) and repeating the same steps we obtain the additive term

$$\begin{aligned}
& \gamma_e(p_z - \frac{\hbar q'_z}{2}) = \sum_{\mathbf{q}''} (n_{\mathbf{q}''} + 1) |F_G(\mathbf{q}'')|^2 \pi \hbar \\
& \delta \left(\epsilon \left(p_z - \frac{\hbar q'_z}{2} \right) - \epsilon \left(p_z - \frac{\hbar q'_z}{2} - \hbar q''_z \right) - \hbar \omega \right)
\end{aligned}$$

which appears in the bracket on the left hand side of (5⁺). By denoting $\gamma = 2(\gamma_a + \gamma_e)$ we are ready to formulate the truncated equation for the first FOD element:

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} + \frac{p_z - \frac{\hbar q'_z}{2}}{m} \nabla_z + eE \nabla_{p_z} + \gamma(p_z - \frac{\hbar q'_z}{2}) + i\omega \right) f_w(z, p_z - \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}, t) = \\
& F_G^*(\mathbf{q}') e^{-iq'_z z} \sqrt{n_{\mathbf{q}'} + 1} \left(f_w(z, p_z - \hbar q'_z, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, \{n_{\mathbf{q}}\}_{\mathbf{q}'}^+, t) - f_w(z, p_z, \{n_{\mathbf{q}}\}, \{n_{\mathbf{q}}\}, t) \right)
\end{aligned} \tag{33}$$

B. Closure of the equation for f_{FOD}^-

The right hand side of equation (5⁻) refers to SOD elements, whose number of phonons in mode \mathbf{q}' of the left basis is reduced by one. These elements are denoted by $f_{SOD}^{-,+}$, $f_{SOD}^{-,-}$, $f_{SOD}^{-,+}$ and $f_{SOD}^{-,+}$ according to the order of their appearance the right. Diagonal elements are provided by the first and the third terms $f_{SOD}^{-,+}$ and $f_{SOD}^{-,-}$. Accordingly, we pursue the equations of motion of the corrections $\delta f_{SOD}^{-,+}$ and $\delta f_{SOD}^{-,-}$ for appropriate con-

tributions to (5⁻). The equations for the rest two terms, namely $f_{SOD}^{-,-}$ and $f_{SOD}^{-,+}$, are also explored for such contributions. The analysis follows the same steps of retaining the FOD elements in the right hand side of the corresponding equations and applying the RPA. Next the obtained contributions are approximated by taking the classical limit. We skip the long but straightforward algebraic steps and summarize the added to the Liouville operator in (5⁻) terms by the following scheme:

$$f_{SOD}^{-,+} \rightarrow \gamma_e(p_z + \frac{\hbar q'_z}{2}); \quad f_{SOD}^{-,-} \rightarrow \gamma_a(p_z + \frac{\hbar q'_z}{2})$$

$$f_{SOD}^- \rightarrow \gamma_a(p_z + \frac{\hbar q'_z}{2}); \quad f_{SOD}^- \rightarrow \gamma_a(p_z + \frac{\hbar q'_z}{2})$$

Collecting all terms into $\gamma = 2(\gamma_a + \gamma_e)$, we formulate the truncated equation for the second FOD element:

$$\left(\frac{\partial}{\partial t} + \frac{p_z + \frac{\hbar q'_z}{2}}{m} \nabla_z + eE \nabla_{p_z} + \gamma(p_z + \frac{\hbar q'_z}{2}) - i\omega \right) f_w(z, p_z + \frac{\hbar q'_z}{2}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}, t) =$$

$$F_G(\mathbf{q}') e^{iq'_z z} \sqrt{n_{\mathbf{q}'}} \left(f_w(z, p_z, \{n_{\mathbf{q}}\} \{n_{\mathbf{q}}\}, t) - f_w(z, p_z + \hbar q'_z, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, \{n_{\mathbf{q}}\}_{\mathbf{q}'}, t) \right)$$

C. The Inhomogeneous Barker-Ferry Equation

Equations (33) and (34) generalize (5). Aiming at a closed equation for the electron Wigner function we follow essentially the same steps used to obtain the Levinson equation.

The corresponding integral equations resemble (7) corrected by the exponents $e^{-\int_{t'}^t \gamma(p_z(\tau) \mp \frac{\hbar q'_z}{2}) d\tau}$ which appear on the right hand sides. A replacement into (4) gives rise to a generalization of equation (8) where $e^{-\int_{t'}^t \gamma(p_z(\tau) - \frac{\hbar q'_z}{2}) d\tau}$ is attached to the exponents of the second row and $e^{-\int_{t'}^t \gamma(p_z(\tau) + \frac{\hbar q'_z}{2}) d\tau}$ is attached to the exponents of the last row. In the latter we switch the sign of \mathbf{q}' , introduce the variable p'_z and perform the trace operation. However, now, the evaluation of the trace is not as straightforward as in the Levinson case. The reason is that γ brings a non-linear dependence from the phonon degrees of freedom so that (11) can be applied after additional considerations of the phonon system. We postpone the discussion of this for the next section and formulate the obtained inhomogeneous Barker-Ferry model:

$$\left(\frac{\partial}{\partial t} + \frac{p_z}{m} \nabla_z + eE \nabla_{p_z} \right) f_w(z, p_z, t) =$$

$$\sum_{\mathbf{q}'_{\perp}, p'_z} \int_0^t dt' \left\{ S(p_z, p'_z, \mathbf{q}'_{\perp}, t, t') f_w(z^-(t'), p'_z(t'), t') \right.$$

$$\left. - S(p'_z, p_z, \mathbf{q}'_{\perp}, t, t') f_w(z^-(t'), p_z(t'), t') \right\}$$

$$S(p_z, p'_z, \mathbf{q}'_{\perp}, t, t') = 2|F_G(\mathbf{q}')|^2 e^{-\int_{t'}^t \bar{\gamma} \left(\frac{p_z + p'_z}{2}(\tau) \right) d\tau}$$

$$\left[n(\mathbf{q}) \cos \left(\int_{t'}^t \frac{(\epsilon(p_z(\tau)) - \epsilon(p'_z(\tau)) - \hbar\omega_{\mathbf{q}'})}{\hbar} d\tau \right) + \right.$$

$$\left. (n(\mathbf{q}) + 1) \cos \left(\int_{t'}^t \frac{(\epsilon(p_z(\tau)) - \epsilon(p'_z(\tau)) + \hbar\omega_{\mathbf{q}'})}{\hbar} d\tau \right) \right]$$

$$\bar{\gamma}(p) = \sum_{\mathbf{q}''} 2\pi\hbar |F_G(\mathbf{q}'')|^2$$

$$[(n(\mathbf{q}'') + 1) \delta(\epsilon(p) - \epsilon(p - \hbar q''_z) - \hbar\omega)$$

$$+ n(\mathbf{q}'') \delta(\epsilon(p) - \epsilon(p + \hbar q''_z) + \hbar\omega)]$$

We note that the Boltzmann out-scattering rate $\bar{\gamma}$ is formally obtained from γ by replacing the phonon coordinates $n_{\mathbf{q}''}$ by their expectation value $n(\mathbf{q}'')$.

V. PHYSICAL ASPECTS

A. The two models

We consider the derivation of the two models from a heuristic point of view. A basic step is the truncation of the GWF hierarchy at the first or the second off-diagonal level. A reasonable argument for such truncation is the assumption for a weak electron-phonon coupling. That is, F , multiplied by the corresponding momentum scale and the time scale β_t of the transitions between the GWF elements must be small. Assumed is an initially decoupled electron-phonon system, where all off-diagonal elements are zero. A typical Levinson transition links a diagonal element to a first off-diagonal element which is linked back to a diagonal element. Two transitions related to the left phonon basis are shown on Fig. 1. A single phonon mode \mathbf{q}' is involved in both links. After the transition, the electron-phonon system returns back to the diagonal state. The assumption for a weak coupling means that only single events occur at given time. Events where two or more transitions involving two or more phonon modes occur at the same time are highly un-probable. Such events are neglected along with events involving higher off-diagonal elements in the path diagonal \rightarrow diagonal element. As inferred by the integral form of (12), the duration of such transition is given by the time interval $t - t'$ in (13). The evolution of the system proceeds on a different time scale β_e which is larger than β_t . This time scale is relevant to the duration T of the measurement of the state of the system.

The Levinson transport picture can be established if the system is mainly decoupled during the process of measurement, i.e. the cumulative time of all transitions is a very small part of the averaging time T : $\beta_e \gg \beta_t$. This is in accordance with the assumption (9) for an equilibrium phonon system (Bloch assumption) which associates a vast mechanism recovering the phonon equilibrium between the transitions.

A nice feature of the Levinson equation is that the classical limit of (13) recovers the Boltzmann equation. This feature provokes a model which explains the evolution in a wide time range in terms of Levinson transitions. In the long time limit these transitions become the instantaneous Boltzmann scattering events. However this model can not be true: A mechanism, which keeps the transitions bounded in time is lacking in the Levinson equation. Indeed, the scattering function S , (13), is not vanishing for large transition times $(t - t')$ so that the time of a transition can become of order of the evolution time, $\beta_t \simeq \beta_e$. From these considerations it is anticipated that the Levinson model can be relevant up to moderate evolution times.

The Barker-Ferry equation enters one level deeper in the hierarchy by accounting the interaction of the FOD with the SOD elements. The main transitions are still of the Levinson type: The diagonal element n, n on Fig. (2) is linked to the diagonal element $n + 1', n + 1'$ via the FOD element $n + 1', n$. Here the prime is for the mode \mathbf{q}' involved in the link. However now the FOD element is modified by the coupling with the SOD elements as shown in the figure. The coupling involves a second mode \mathbf{q}'' and corresponds to instantaneous transitions from FOD to SOD elements and back. These transitions are integrated into the exponent of (36): To see this we compare (7) with (45) and recall the equivalence of the latter with (47), (with λ_1 set to zero). We note that the bounds of the time integral in the exponent correspond to the duration of the main transition, while the classical limit makes the interaction with the SOD elements instantaneous. All such SOD elements are accounted by sum over the second mode \mathbf{q}'' . The Barker-Ferry model already introduces a mechanism which takes care for the duration of the transitions: the exponent effectively damps the long lasting correlations. However in the long time limit the model fails to recover the classical delta function, giving rise to a Lorentzian energy distribution instead.

These considerations suggest the existence of a time limit \mathcal{T} of validity of the two models. The correlations with the higher off-diagonal elements can no more be neglected after this time. An estimation of \mathcal{T} , which certainly depends on the physical parameters of the system, can be obtained by numerical experiments. A credibility guess for \mathcal{T} is the time when the behavior of the observables begins to become unphysical. As already noted, the Lorentzian gives rise to artificial carrier heating, so that the total energy of the system is one of the candidate observables. Is the carrier density also such an

observable? An integration of (12) or (35) with respect to p_z shows that the two models give rise to the continuity equation. Hence if the initial condition is physically relevant one can't expect surprises in the evolution of the carrier density such as appearance of negative carrier concentrations. However nothing can be set in advance for the evolution of the momentum density. While in the classical case the non-negative Boltzmann scattering rate guarantees a physical momentum distribution, the rates (13) and (36) allow negative values. Thus the behavior of the momentum density at longer evolution times could be another candidate for probing the validity of the models.

B. The phonon system

The appearance of the Boltzmann out-scattering rate $\bar{\gamma}$ in (37) can be established in two ways. One can follow the quite common approach of replacement of complicatedly behaving functions by appropriate their averages. As the phonons are assumed in equilibrium, the probability to find $n_{\mathbf{q}}$ phonons in mode \mathbf{q} is $P(n_{\mathbf{q}})$. By recalling the dependence of γ on $\{n_{\mathbf{q}}\}$ it sufficient to average γ to obtain:

$$\bar{\gamma} = \sum_{\{n_{\mathbf{q}}\}} \prod_{\mathbf{q}} P_{eq}(n_{\mathbf{q}}) \gamma \quad (38)$$

Further on $\bar{\gamma}$ is replaced into the generalized equation (8) and the steps associated with (9)-(11) are applied afterwards.

There is an alternative approach, which throws more insight in the assumptions hidden in (38). In the generalized equation (8) we first expand $e^{-\int_{t'}^t \gamma}$ into a series and truncate to a term K corresponding to a desired precision. Then we apply the trace operation together with the Ansatz (9). Obtained are terms of the form:

$$\sum_{\{n_{\mathbf{q}}\}} n_{\mathbf{q}_0} \sum_{\mathbf{q}_1} n_{\mathbf{q}_1} \sum_{\mathbf{q}_2} (n_{\mathbf{q}_2} + 1) \dots P_{eq}(n_{\mathbf{q}_0} + I) \quad (39)$$

$$\prod_{\mathbf{q} \neq \mathbf{q}_0} P_{eq}(n_{\mathbf{q}}) |F_G(\mathbf{q}_1)|^2 |F_G(\mathbf{q}_2)|^2 \dots \delta_{q_{z1}} \delta_{q_{z2}} \dots$$

Here the sum of the trace runs over the natural numbers from zero to infinity for any mode. As shown below, we can take the average on any phonon mode \mathbf{q} independently on the rest of the modes in (39). That is, under a reasonable assumption, terms with repeating modes of the form $n_{\mathbf{q}} n_{\mathbf{q}} \dots P_{eq}(n_{\mathbf{q}})$ can be neglected. The first factor $n_{\mathbf{q}_0}$ corresponding to \mathbf{q}' in (8) is distributed according $P_{eq}(n_{\mathbf{q}_0} + I)$ where I is ± 1 or 0 . With the help of (11) it is evaluated to $n(\mathbf{q}_0)$ or $n(\mathbf{q}_0) + 1$, and can be skipped in the further discussions. The modes $\mathbf{q}_1 \mathbf{q}_2 \dots$ come from the product $\gamma \gamma \dots$ in the expansion of the exponent. The phonons in these modes are distributed according $P_{eq}(n_{\mathbf{q}})$. After averaging, the equilibrium phonon numbers $n(\mathbf{q}_i)$ replace the corresponding

phonon coordinates $n_{\mathbf{q}_i}$ in (39). These terms sum up recovering the exponent, which settles the appearance of $\bar{\gamma}$ in (36).

Now we show that, provided that the number of the modes in the plane normal to the wire is large enough, the trace can be taken by neglecting the terms with repeating modes in the product (39). The delta functions in γ affect only the z coordinates of the wave vectors as indicated by the index in $\delta_{q_{z1}}$. They can be accounted, and (39) can be further decomposed into products with fixed combinations of z coordinates $q_{z1}q_{z2}\dots q_{zk}$:

$$\sum_{n_{\mathbf{q}_1}\dots n_k=0}^{\infty} \sum_{\mathbf{q}_{\perp 1}} n_{\mathbf{q}_1} \dots \sum_{\mathbf{q}_{\perp k}} n_{\mathbf{q}_k} P_{eq}(n_{\mathbf{q}_1}) \dots P_{eq}(n_{\mathbf{q}_k}) |F_G(\mathbf{q}_1)|^2 \dots |F_G(\mathbf{q}_k)|^2$$

We note that modes where F_G is zero (e.g. in the zeros of the electron state in the normal plane) can be skipped in the sum. We assume that the number of these modes are negligible as compared to the number N of the modes in the normal plane (normal modes). Consider the product:

$$\sum_{n_{\mathbf{q}_1}\dots n_{\mathbf{q}_k}=0}^{\infty} n_{\mathbf{q}_1} \dots n_{\mathbf{q}_k} P_{eq}(n_{\mathbf{q}_1}) P_{eq}(n_{\mathbf{q}_2}) \dots P_{eq}(n_{\mathbf{q}_k}) \quad (40)$$

This expression is evaluated to

$$n(\mathbf{q}_1) \dots n(\mathbf{q}_k); \quad \text{if} \quad \mathbf{q}_1 \neq \mathbf{q}_2 \neq \dots \neq \mathbf{q}_k \quad (41)$$

or, if some modes coincide to:

$$n(\mathbf{q}_1) \dots n^{(\alpha)}(\mathbf{q}_j) \dots n^{(\beta)}(\mathbf{q}_m) \dots n(\mathbf{q}_k);$$

$$n^{(\alpha)}(\mathbf{q}) = \sum_{n_{\mathbf{q}}=0}^{\infty} n_{\mathbf{q}}^{\alpha} P_{eq}(n_{\mathbf{q}}) \quad (42)$$

Now let l be the infimum and m_k the supremum defined as:

$$L = \inf_{\mathbf{q}} n(\mathbf{q}) |F_G(\mathbf{q})|^2$$

$$M_k = \sup_{\alpha\beta\dots; \mathbf{q}_0\dots\mathbf{q}_k} n(\mathbf{q}_1) \dots n^{(\alpha)}(\mathbf{q}_j) \dots \dots n^{(\beta)}(\mathbf{q}_m) \dots n(\mathbf{q}_k); |F_G(\mathbf{q}_1)|^2 \dots |F_G(\mathbf{q}_k)|^2$$

These numbers exist due to the discrete character of the modes. Moreover l is different from zero since the zeros of F_G are already skipped. We evaluate the relative contributions of the terms (42) to the sum (40). As the number of the terms (41) is $N(N-1)\dots(N-k+1)$ (we assume the worst case scenario of equal z coordinates) this contribution is less than:

$$\frac{M_k(N^k - N(N-1)\dots(N-k+1))}{L^k N(N-1)\dots(N-k+1)}$$

The latter tends to zero if N tends to infinity for any fixed $k \leq K$. It is concluded that if the number of the normal modes is very large we can keep only the terms

(41). This result concerns also the Bloch assumption: An initially equilibrium phonon system can be considered as equilibrium at later times provided that the number of modes is large enough. In this case the assumption for a vast mechanism of phonon relaxation is no more indispensable.

VI. RESULTS

The implemented numerical approach is a Backward Monte Carlo method. The method has been first developed³³ and refined^{34,35} for the case of classical transport. The application of the method to quantum transport equations has been initiated in the last decade^{21,36}. The idea of the approach is to express the solution of the equation into a Neumann series and to evaluate the consecutive terms with the help of Markov chains. The latter are constructed by consecutive application of an a priori selected transition probability. The Markov chains begin from the fixed point and time where the value of the solution is to be determined. The chains "evolve" backward in time, which gives the name of the approach. In general the precision depends on the number the chains N as $N^{-1/2}$. High precision is obtained by increasing the number N , which is on the expense of a corresponding increase of the computational time.

The approach is readily generalized for evaluation of functionals of the solution. In the indirect way the solution is computed in selected points which are used in the chosen quadrature for computation of the functional. In the direct way the functional is computed with the help of an initial probability used to select the initial points of the Markov chains.

A *GaAs* quantum wire with a square cross section of 10 nanometers is chosen for the simulations. The material parameters are taken from²¹, in particular a single polar optical phonon having a constant energy $\hbar\omega$ is considered. The electric field is zero. The initial condition is a product of two Gaussian distributions of the energy and space. The k_z^2 distribution corresponds to a generating laser pulse with an excess energy of about $150meV$. The z distribution is centered around the origin of the coordinate system. We first regard the evolution of the wave vector (and similar the energy) and the density distributions given by the integrals

$$f(k_z, t) = \int \frac{dz}{f_w} (z, \hbar k_z, t); \quad n(z, t) = \int \frac{dp_z}{f_w} (z, p_z, t)$$

Here it is assumed that the initial condition is normalized to unity. The behavior of the energy distribution at very low temperatures can be used as a test of the correctness of our approach: as the wire electrons remain in the ground state in the normal plane the peculiarities of the evolution in the wire must be the same as in the homogeneous case. We recall the major features of the homogeneous evolution¹⁴. Semiclassical electrons

can only emit phonons and loose energy equal to a multiple of the phonon energy $\hbar\omega$. They evolve according to an energy distribution, patterned by replicas of the initial condition shifted towards low energies. Such electrons cannot appear in the region above the initial distribution. Since the lack of the energy conserving delta function the quantum solutions demonstrate two effects of deviation from the semiclassical behavior. The replicas are broadened and the broadening reduces with the time. A finite density of electrons appears in the semiclassical forbidden region above the initial condition. The wire electrons show the same behavior. The function $f(k_z, t)$ is symmetric with respect to the origin and thus $f(k_z > 0, t)$ is a representative for the behavior of the energy distribution. Fig.(3) shows the initial condition and the well broadened curve $f(|k_z|, t)$ at time $t = 50$ femtoseconds of the evolution. The presence of electrons above the initial condition is visible at around $k_z = 60[10^{-2}/nm]$. At such small times there is no difference between the solutions of the Levinson and the Barker-Ferry models. The broadening begins to shrink with the increase of the time, Fig. (4), the first peak to the left of the initial condition is already formed after 150 femtoseconds of evolution and the second one comes up. The distance between the initial and the first peaks corresponds to a shift with the phonon energy, however the curve is still wider as compared to an exact replica. The 175 femtoseconds curve is obtained by a direct evaluation of the functional for the wave vector density. The curves at earlier times are computed indirectly via the Wigner function evaluated in $800 \times 260 z$ and kz mesh-points. The points are regularly distributed in the simulation domain with steps of $1nm$ and $0.5[10^{-2}/nm]$ respectively. With the increase of the evolution time the indirect way becomes more and more inexact. Fig. 5 compares the 175 femtoseconds densities obtained by direct and indirect computations from the Levinson model. The indirect curve is already unphysical. The result is independent on how precisely the Wigner function is evaluated in the chosen points provided that the mesh is kept fixed. This feature is associated to the fact that with the increase of the evolution time the Wigner function, Fig. 6, becomes less and less smooth function and thus an increased number of mesh-points are needed for a precise evaluation of the corresponding functionals. The Barker-Ferry curve plotted with the solid line in Fig. 5 is obtained by the direct way. A comparison with the Levinson counterpart shows that the exponent has a pronounced effect on the wave vector distribution. It causes an effective retardation to the evolution process. The difference between the two models is expected to become more pronounced at larger times. Unfortunately the computational burden increases exponentially with the evolution time. Special numerical approaches are needed to achieve few hundreds of femtoseconds, which is a task beyond the aims of this work.

Fig. X shows the initial electron density (dashed line) centered around the origin. It splits into two peaks which

move in the positive and negative directions of the wire. The 50 femtoseconds density is presented by the solid line. A comparison with the ballistic curve (dotted line) shows that at early times the spatial transport is mainly ballistic. An exception is the central part, already filled with electrons slowed down by the interaction with the phonons. The situation changes entirely with the increase of the time. Fig. 8 compares the ballistic density with the densities obtained from the two models. The fronts of the ballistic peaks, placed above (below) $200nm$ ($-200nm$) are formed by the fastest classical electrons since processes of phonon absorption are suppressed at $T = 0K$. The quantum fronts are placed further away of the origin. They are formed by electrons which gained velocity from the interactions: such electrons reside in the classically forbidden energy region and thus move faster. This picture is asserted by the distribution of the mean energy per particle

$$e(z, t) = \int dk_z \epsilon(k_z) f(z, k_z, t) / n(z, t)$$

Fig. 9 shows $e(z, t)$ for $t = 175fs$. The initial distribution gives a reference for the energy range of the ballistic electrons. Outside the $200nm$ region around the origin the quantum electrons are much hotter than the ballistic ones. The difference of the energy distribution provided by the two models is already well pronounced in the central part. The Barker-Ferry curve is closer to the initial distribution which is in accordance with the delay in the evolution of the replicas caused by the exponential damping of the interaction durations.

VII. CONCLUSIONS

The generalized Wigner function provides a convenient instrumentality for derivation of quantum-kinetic models of the electron-phonon interaction. The corresponding hierarchy of Wigner function elements can be truncated at different levels giving rise to closed equations for the electron system. Derived are the inhomogeneous counterparts of the Levinson and Barker-Ferry equations, which describe the femtosecond evolution of local electron packets in a quantum wires. Basic are the hypotheses for an initially decoupled system, equilibrium phonons and the Markov approximation. The physical aspects of the set of the incorporated assumptions is discussed. In particular it is argued that the relevance of both models is bounded at the long time limit. The solutions of the equations are rich of quantum effects already at the case of zero electric field. Along with the collisional broadening and retardation, an effect of ultrafast spatial transport is observed. This effect has been reported recently for the case of infinite electron lifetime¹⁵. The solutions of the two models begin to differ after around 200 femtoseconds of evolution. The next few hundreds of femtoseconds is the most interesting time domain for analysis of the features of the two models. Unfortunately the numerical burden

increases rapidly with the evolution time. Novel numerical approaches, including GRID technologies, aiming to explore this time domain along with the effect of the electric field are currently under development.

VIII. ACKNOWLEDGMENTS

IX. APPENDIX A

The integral form of equation (2) is obtained with the help of the characteristics of the Liouville operator on

$$f_w(z, p_z, \cdot, t) = f_0(z(0), p_z(0), \cdot, 0) e^{-i/\hbar(\epsilon(\{n_{\mathbf{q}}\}) - \epsilon(\{n'_{\mathbf{q}}\}))t} + \int_0^t dt' e^{-i/\hbar(\epsilon(\{n_{\mathbf{q}}\}) - \epsilon(\{n'_{\mathbf{q}}\}))(t-t')} \times \quad (44)$$

$$\sum_{\mathbf{q}'} F(\mathbf{q}') \left\{ G(\mathbf{q}'_{\perp}) e^{iq'_z z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q'_z}{2}, \cdot, t') - G^*(\mathbf{q}'_{\perp}) e^{-iq'_z z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z, p_z(t') + \frac{\hbar q'_z}{2}, \cdot, t') \right.$$

$$\left. - G(\mathbf{q}'_{\perp}) e^{iq'_z z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z, p_z(t') + \frac{\hbar q'_z}{2}, \cdot, t') + G^*(\mathbf{q}'_{\perp}) e^{-iq'_z z(t')} \sqrt{n_{\mathbf{q}'} + 1} f_w(z(t'), p_z(t') - \frac{\hbar q'_z}{2}, \cdot, t') \right\}$$

The integral form can be proved by taking the time derivative of 44 which should lead us to 2. To see this we rewrite 44 by keeping only the relevant variables:

$$f(z, p_z, t) = f(z(0), p_z(0), 0) e^{\lambda t} + \quad (45)$$

$$\int_0^t dt' K(z(t'), p_z(t')) e^{\lambda(t-t')} f(z(t'), p_z(t'), t')$$

which can be augmented to:

$$f(z(t''), p_z(t''), t'')|_{t''=t} = f(z(0), p_z(0), 0) e^{\lambda t''}|_{t''=t} +$$

$$\int_0^{t''} dt' K(z(t'), p_z(t')) e^{\lambda(t''-t')} f(z(t'), p_z(t'), t')|_{t''=t}$$

Written in this way, the equation reminds that the time derivative is taken over the trajectory: first we differentiate with respect to t'' and set $t'' = t$ in the final result. The left hand side readily gives the Liouville operator \mathcal{L} acting on f , while the right hand side gives $\lambda f(z, p_z, t) + K(z, p_z) f(z, p_z, t)$, (compare 2)

$$(\mathcal{L} - \lambda) f(z, p_z, t) = K(z, p_z) f(z, p_z, t) \quad (46)$$

Equation (45) can be formulated in an alternative way provided that $\lambda = \lambda_1 + \lambda_2$:

$$f(z, p_z, t) = f(z(0), p_z(0), 0) e^{\lambda_1 t} + \quad (47)$$

$$\int_0^t dt' (K(z(t'), p_z(t')) + \lambda_2) e^{\lambda_1(t-t')} f(z(t'), p_z(t'), t')$$

The equivalence between (45) and (47) will be used in finding models which approximate equation (2). Note

the left, which are Newton's trajectories of the form:

$$z(t') = z - \frac{1}{m} \int_{t'}^t p_z(\tau) d\tau \quad p_z(t') = p_z - eE(t-t'); \quad (43)$$

The particular trajectory is initialized at (z, p_z, t) . The equation includes explicitly the initial condition $f_0(z, p_z, \{n_{\mathbf{q}}\}, \{n'_{\mathbf{q}}\}, 0)$. The phonon coordinates of the particular terms the same in the integral form and the integro-differential counterpart. Referring to this correspondence we keep them implicit in the GWF elements:

that (45) and (47) continue to hold if λ is time dependent: then the argument of the exponents (of the form $\lambda \cdot (t-t_1)$, $t_1 = 0, t'$) must be replaced by $\int_{t_1}^t \lambda(\tau) d\tau$. We also note the equivalence between the expressions $p_z(t') + \hbar q_z$ and $(p_z + \hbar q_z)(t')$.

X. APPENDIX B

Denoting $f_w(\cdot, t')$ shortly by $\phi(t')$ and introducing relative time variables $t_1 = t' - t''$, $\tau_1 = \tau - t''$ we rewrite the integral of interest I as follows:

$$I = \int_{t''}^t dt' e^{i \int_{t''}^{t'} \frac{(\Delta\epsilon(\tau) + \hbar\omega) d\tau}{\hbar}} \phi(t') =$$

$$\int_0^{t-t''} dt_1 e^{i \int_0^{t_1} \frac{(\Delta\epsilon(\tau_1 + t'') + \hbar\omega) d\tau_1}{\hbar}} \phi(t_1 + t'')$$

$$= \int_0^{t-t''} dt_1 e^{i \frac{t_1}{\hbar} \left\{ \Delta\epsilon(t'') + \hbar\omega + \frac{\hbar^2 q''_z e E t_1}{2m} \right\}} \phi(t_1 + t'')$$

The time integration in the last row has been performed with the help of the definition of the trajectory (43) by using the explicit expression for $\Delta\epsilon$. Denote by α and β the scales of the energy and the time respectively. A dimensionless variable $t^\beta = t/\beta$ is introduced. We assume that α is a common scale for the electron and phonon energies and also for the term determined by the electric

field, so that:

$$\epsilon = \epsilon^\alpha \alpha; \quad \epsilon_{ph} = \hbar\omega = \epsilon_{ph}^\alpha \alpha; \quad \epsilon_E = \frac{\hbar^2 q_z \epsilon E \beta}{2m} = \epsilon_E^\alpha \alpha$$

A new integration variable

$$x = \frac{t_1 \alpha}{\hbar} = \frac{t_1^\beta \beta \alpha}{\hbar} = \frac{t_1}{\hbar_\alpha} = \frac{t_1^\beta}{\hbar_{\alpha\beta}}$$

where $\hbar_\alpha = \hbar/\alpha$ and $\hbar_{\alpha\beta} = \hbar/(\alpha\beta)$ is introduced. Furthermore the shortening $T = (t-t'')/\hbar_\alpha = (t-t'')^\beta/\hbar_{\alpha\beta}$ will be used. With the help of these variables the integral is rewritten as:

$$I = \hbar_\alpha \int_0^T dx e^{i\{\Delta\epsilon^\alpha + \epsilon_{ph}^\alpha + \epsilon_E^\alpha x \hbar_{\alpha\beta}\}x} \phi(\beta x \hbar_{\alpha\beta} + t'')$$

We assume that the energy and time scales are large enough to allow us to apply the limit $\hbar_{\alpha\beta} \rightarrow 0$. Then $T \rightarrow \infty$, while the term related to the electric field vanishes. The time integral in I becomes the Fourier transform of the step function $\theta(t)$ which is expressed with the help of generalized functions:

$$I = \hbar_\alpha \int_0^\infty dx e^{i\{\Delta\epsilon^\alpha + \epsilon_{ph}^\alpha\}x} \phi(t'') = \hbar \left(\pi \delta(\Delta\epsilon + \hbar\omega) + v.p. \frac{i}{\Delta\epsilon + \hbar\omega} \right) \phi(t'')$$

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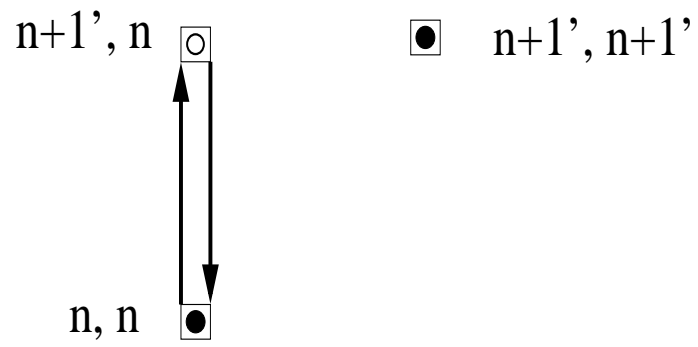
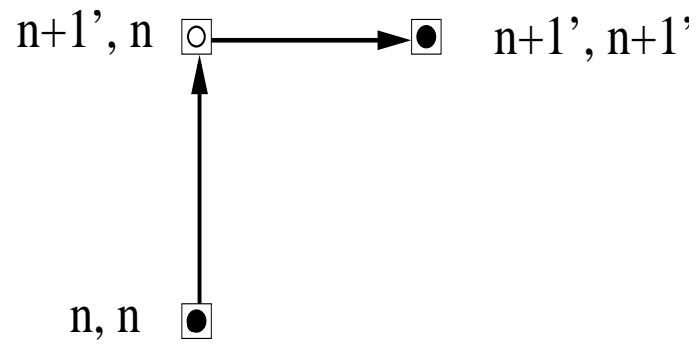


FIG. 1: Levinson transitions link diagonal with FOD elements. The FOD element is presented by the empty circle and n, n stands for the left and right phonon set $\{n_q\}, \{n_q\}$

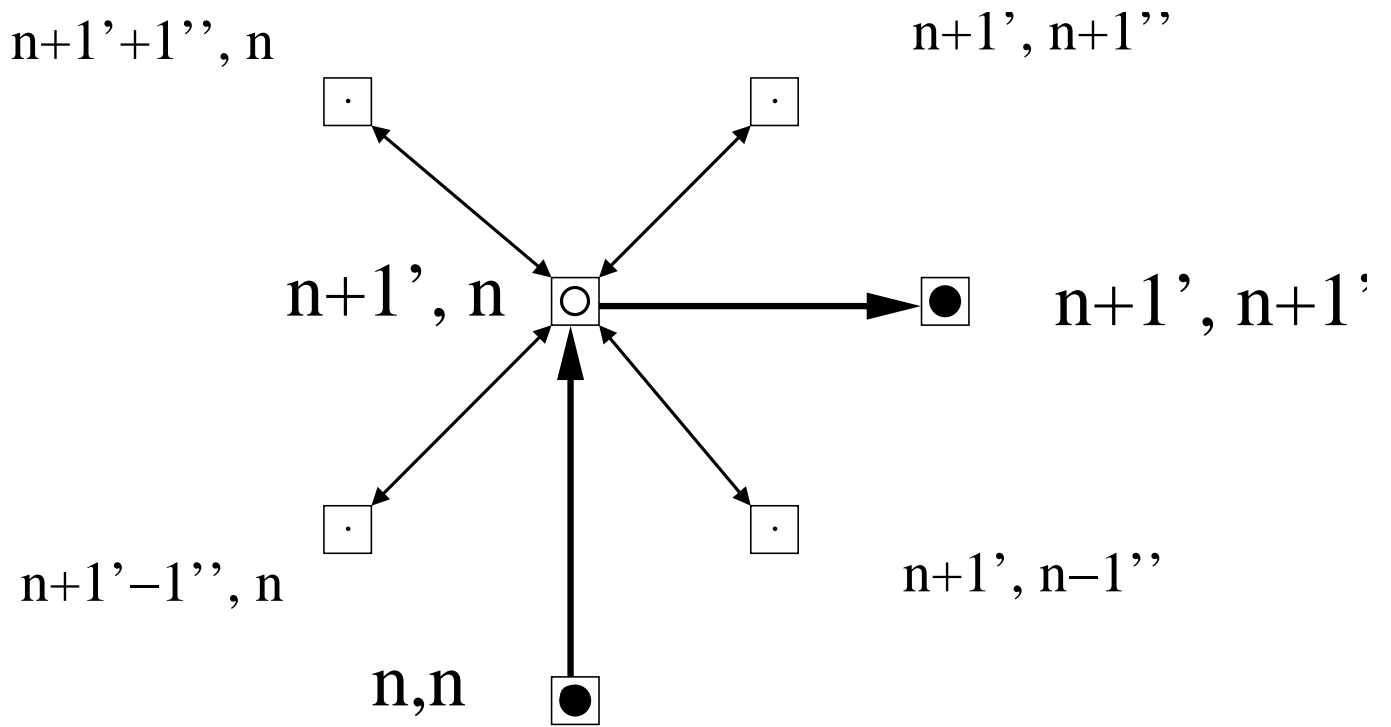


FIG. 2: The Barker-Ferry counterpart of the first Levinson transition shown on Fig. 1. $\pm 1''$ denotes increase or decrease by unity of the phonons in mode \mathbf{q}'' in the left or the right basis.

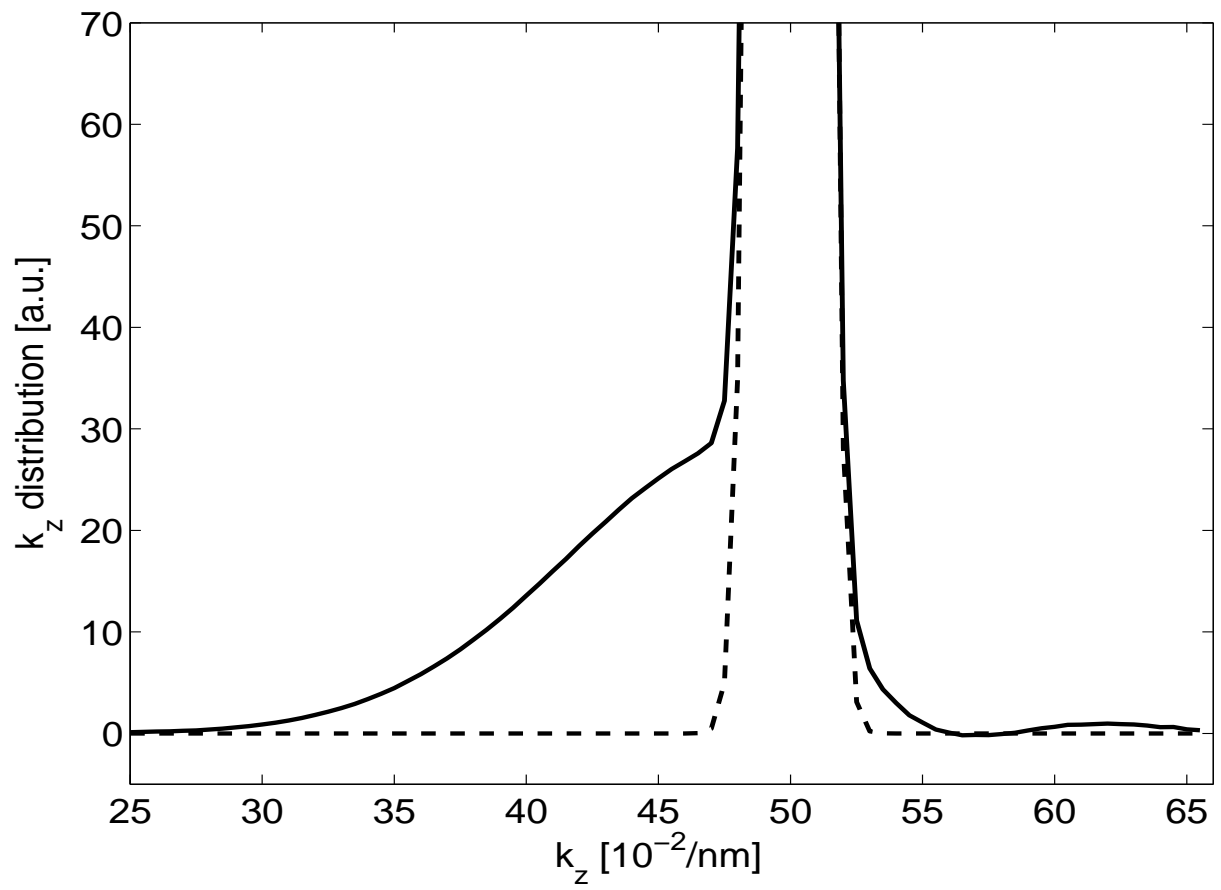


FIG. 3: Initial peak (dashed line) and the 50 femtoseconds wave vector density presented in a window of positive k_z .

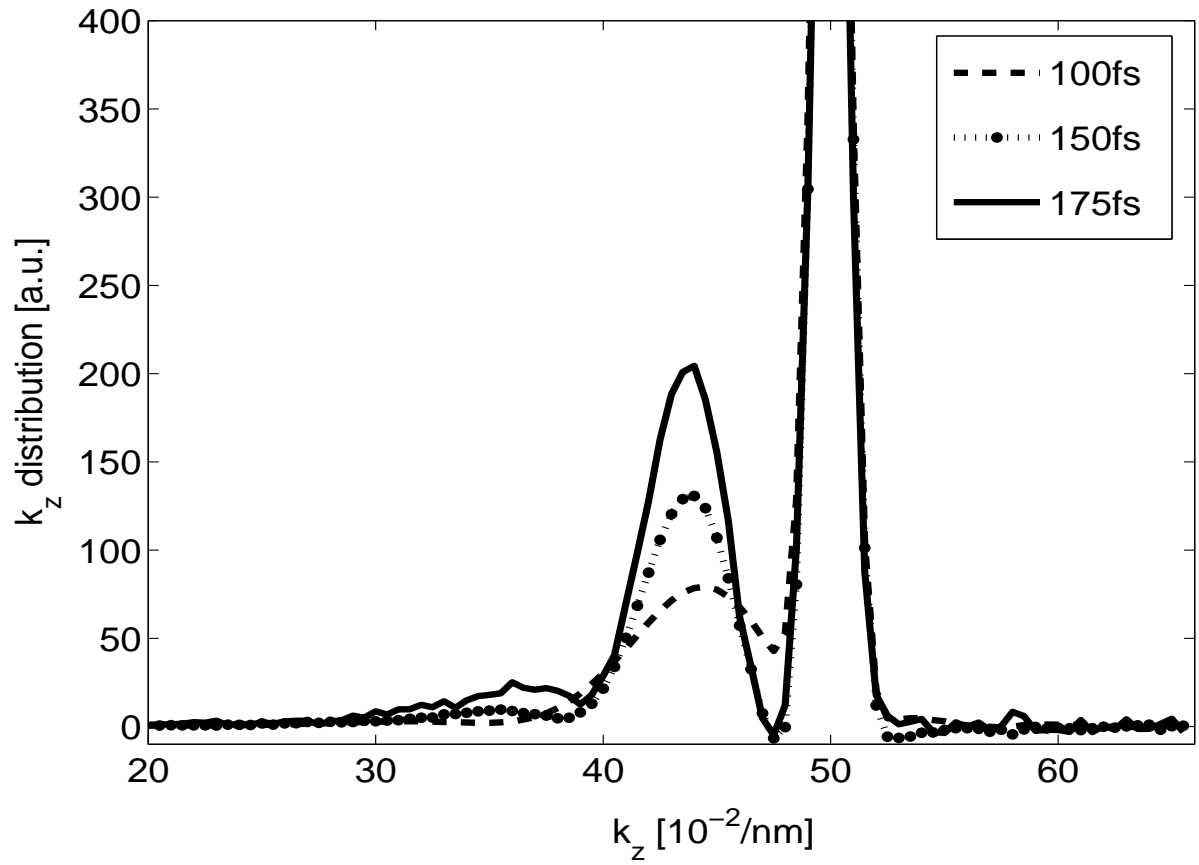


FIG. 4: Time evolution of the wave vector distribution obtained from the Levinson model. The 175 femtoseconds curve is obtained by direct evaluation of $f(k_z, t)$. The distributions at earlier times are computed indirectly, via the Wigner function.

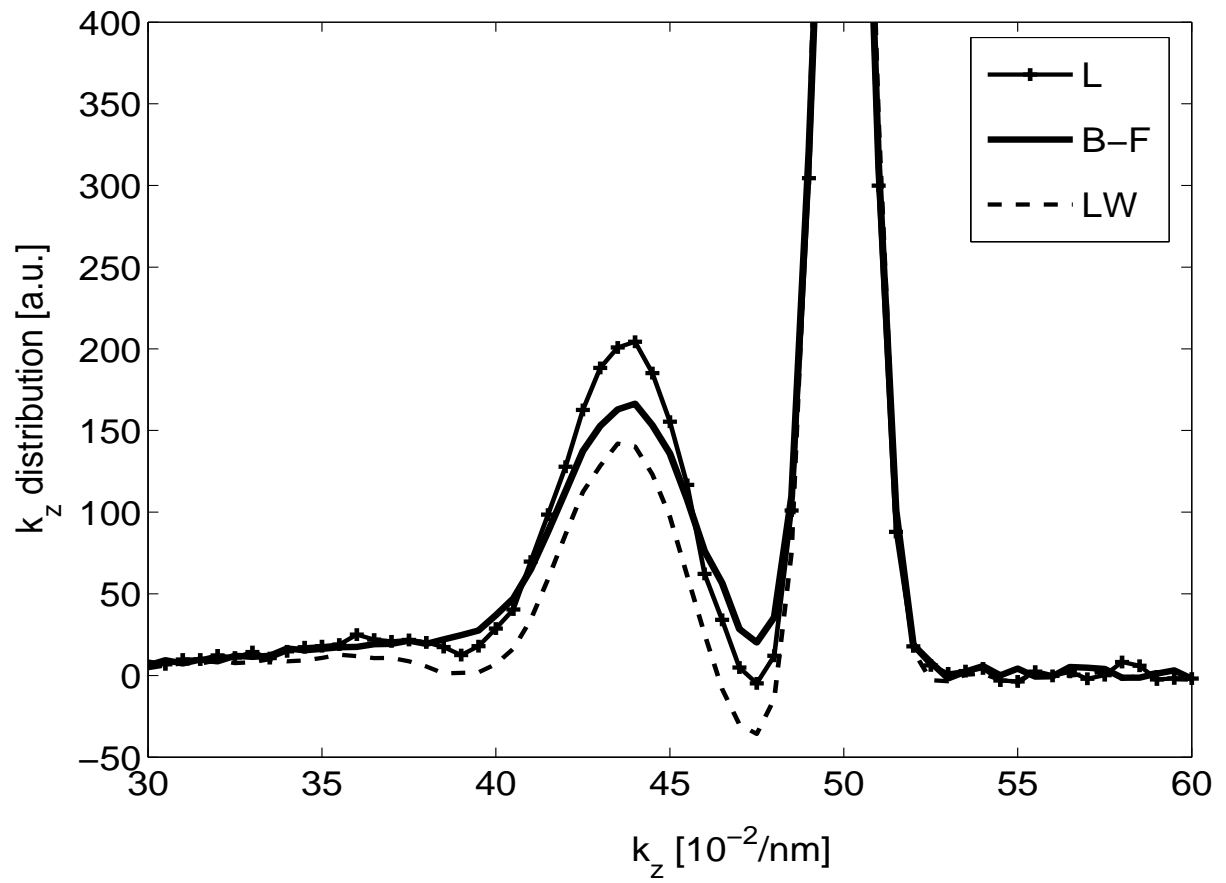


FIG. 5: The 175 femtoseconds wave vector distribution obtained from the Levinson model by direct (L) and indirect (LW) computations via the Wigner function. The difference between the two results is significant, the LW curve becomes negative in the valley between the peaks. The Barker-Ferry curve (solid line) is obtained by direct computations.

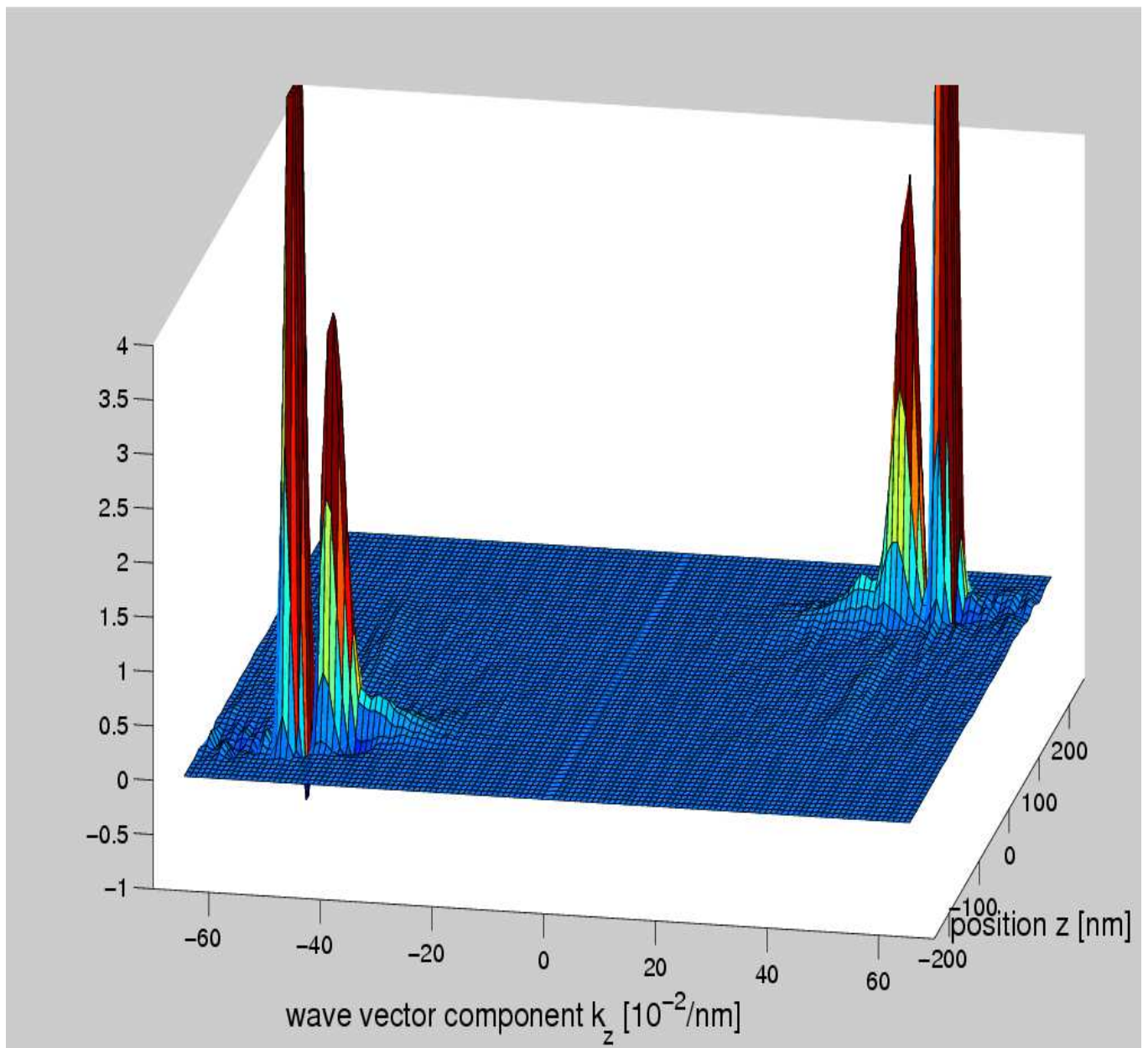


FIG. 6: The Wigner function after 175 femtoseconds of evolution. The two main peaks are truncated for better resolution. The secondary peaks are formed by electrons which loosed energy from the phonons. The classically forbidden regions are placed on the opposite side of the main peaks, in the nearest left and the far right corners of the picture. The regions are populated with vastly moving electrons.

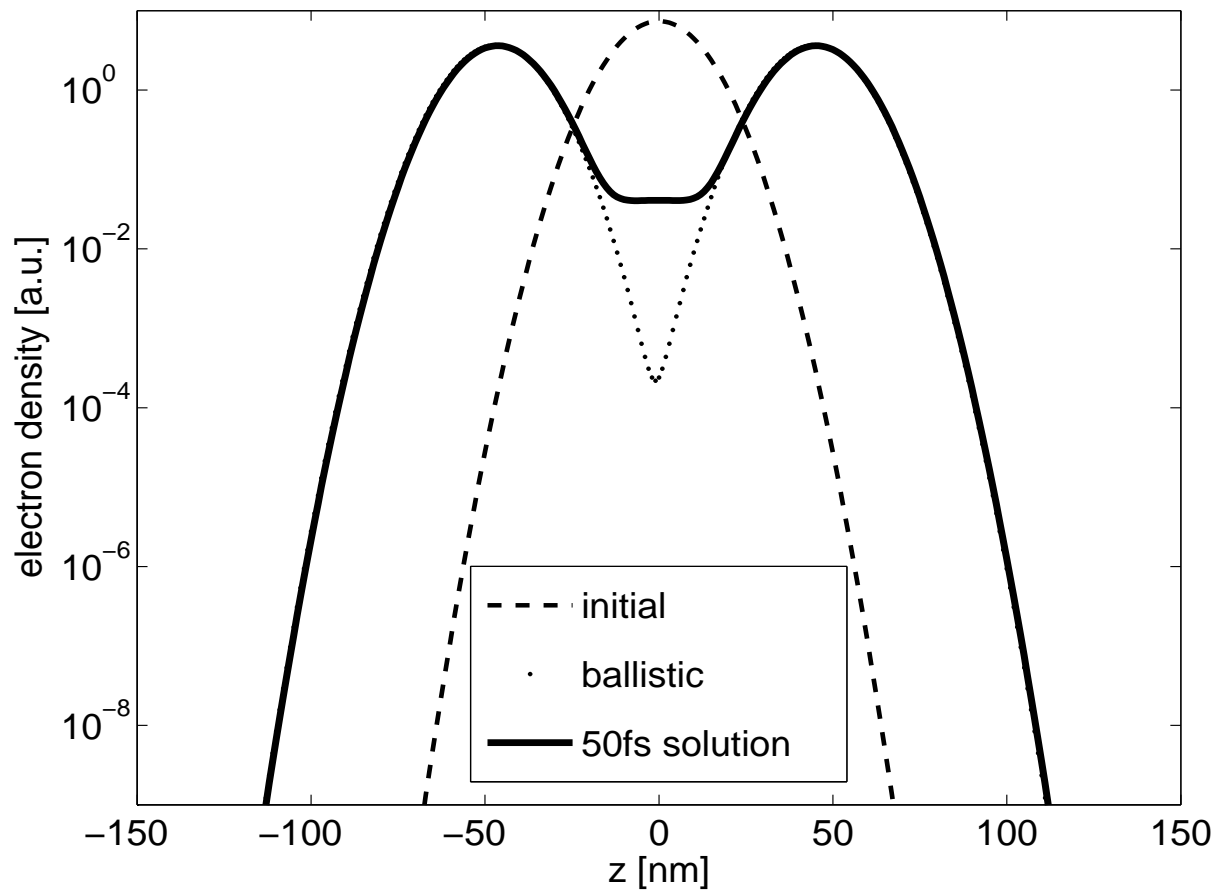


FIG. 7: Electron densities at time 0 and 50 femtoseconds. The ballistic curve (dotted line) coincides everywhere with the 50fs solution but the central part. There reside electrons slowed down by the scattering events.

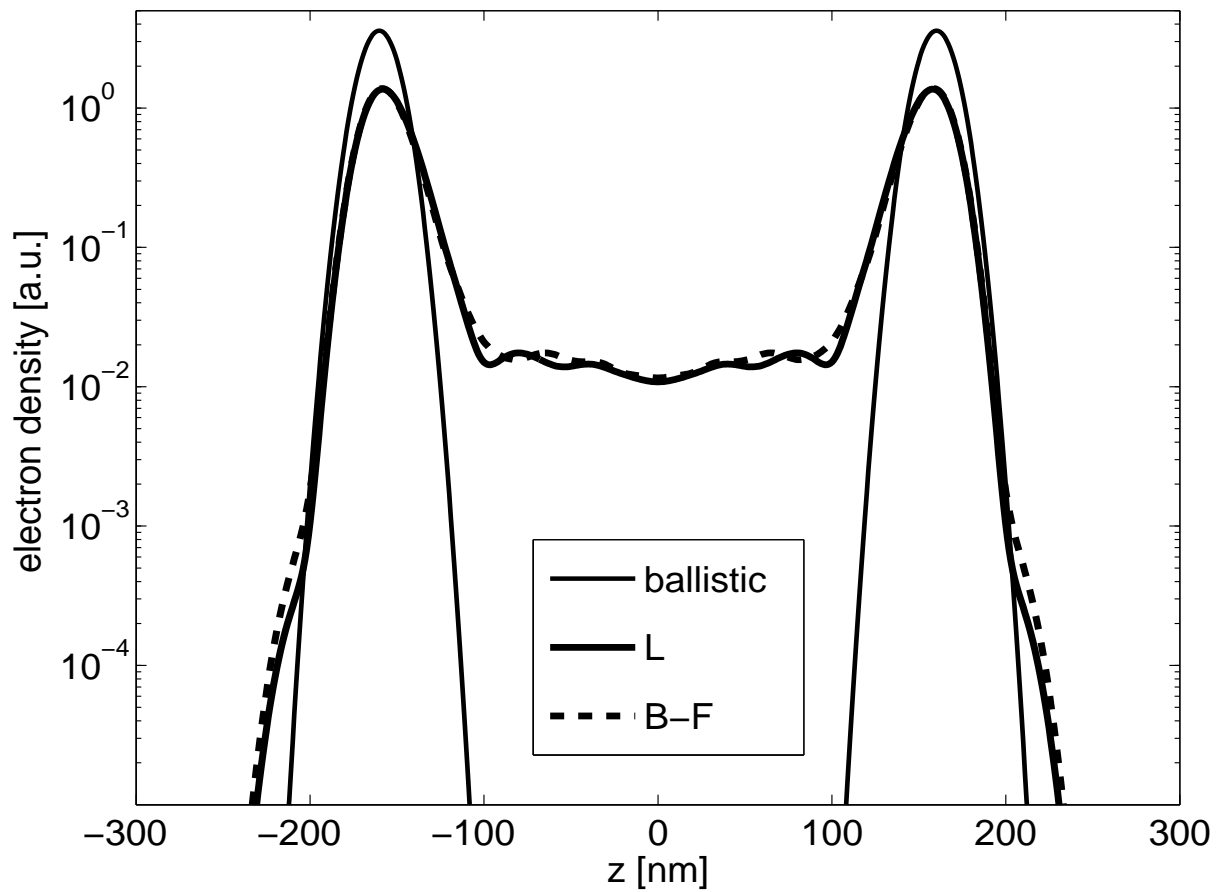


FIG. 8: Electron density after 175 femtoseconds evolution. The fastest classical electrons form the fronts of the two ballistic peaks (solid line) slightly above (below) 200nm (-200nm

. The fastest quantum electrons of the Levinson (bold line) and Barker-Ferry (dashed line) models reach distances placed further away from the origin.

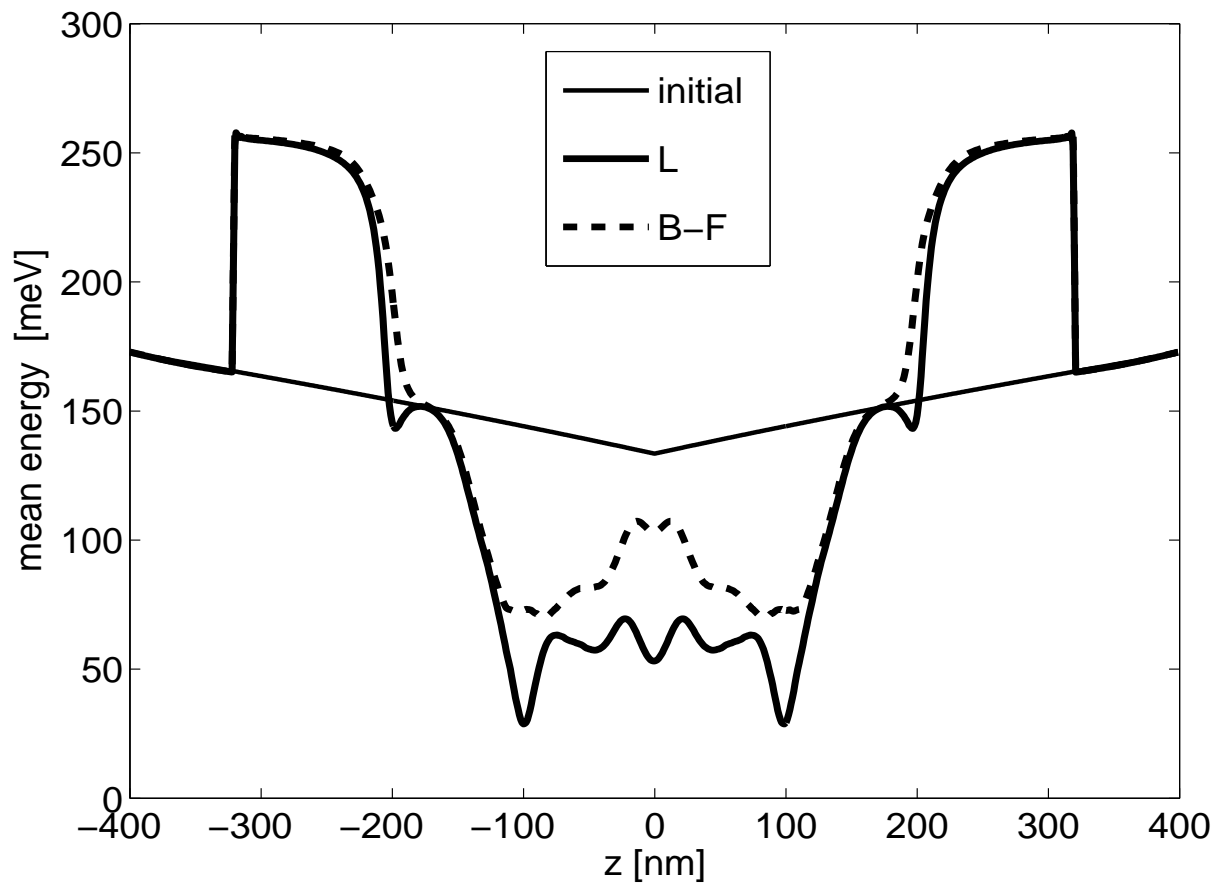


FIG. 9: Initial (solid line) Levinson (bold line) and Barker-Ferry (dashed line) distributions of the mean energy per particle at 175 femtoseconds.