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# Time-dependent order and distribution policies in supply networks

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**Summary.** The dynamic of a production networks is modeled by a coupled system of ordinary differential delay equations. Distribution and order policies are determined by an optimization problem for maximizing the profit of the production line.

## 1 Introduction

We consider a network of suppliers which order goods from each other and process a product according to orders, and receive payments according to a pricing policy. The dynamics of supply chains has been investigated in recent years (see c.f. [1, 2, 3, 5, 7]) and extended to include money flows and bankruptcy, e.g. [2]. We extend existing results in the following ways: We consider general networks, represented by an arbitrarily connected graph. Each node in the network has a finite production or cycle time and a finite production capacity, as well as a front-end and back-end inventory. It is therefore possible that a supplier orders more than can be produced and stockpiles supplies. It is also possible that a supplier produces more than is ordered and stockpiles the output. Each supplier receives payments according to a dynamically determined pricing policy. Bankruptcy occurs if payments made exceed payments received beyond a certain available credit limit. Distribution and order policies are chosen in order to maximize the total profit. Mathematically, this problem is formulated as a mixed–integer programming problem.

## 2 The model

The supply chain is modeled as a network of  $S_1, \dots, S_J$  nodes (suppliers) which order and deliver goods according to given (dynamic or static) policies, and

receive payments according to certain pricing policies. We suppose that each supplier can choose his own policy. We furthermore assume that each supplier  $S_j$  has two inventories, a front-end (or input) inventory with an inventory position  $p_j(t)$ , and a back-end (or output) inventory with inventory position  $q_j(t)$ . Items are taken from the input inventory, processed by a given time  $\tau_j$ , put into the output inventory and instantaneously delivered according to demand. Each supplier cannot take more than  $\mu_j dt$  items in every infinitesimal time interval  $dt$ , i.e., the supplier has a maximal capacity of  $\mu_j$  on the production process. There is no restriction on the inventories. This allows for storing overproduction and buffering for shortages in the supplies. Adjacent suppliers are directly connected by the rate of orders and flow, respectively. That means, supplier  $k$  orders products from supplier number  $j$  at a rate  $\Omega_{jk}$  and supplier  $k$  sends products to supplier number  $j$  at a rate  $\Phi_{jk}$ .

We turn to the mathematical equations describing the above model. We model the input inventory position  $p$  by the simple ordinary differential equation (ODE) and put a constraint such that the inventory cannot become negative.

$$\frac{dp}{dt} = \begin{cases} f^{in} - \mu & \text{for } p > 0 \\ 0 & \text{for } p = 0 \end{cases}. \quad (1)$$

The ODE (1) has a discontinuous right hand side and is therefore not guaranteed to have a solution and we therefore we replace (1) by the smooth dynamics

$$\frac{dp}{dt} = \begin{cases} f^{in} - \mu & \text{for } p > \varepsilon\mu \\ f^{in} - \frac{p}{\varepsilon} & \text{for } p \leq \varepsilon\mu \end{cases} = f^{in} - g, \quad g = \min\{\mu, \frac{p}{\varepsilon}\}, \quad 0 < \varepsilon \ll 1. \quad (2)$$

It is easy to see that (2) exhibits, for small  $\varepsilon$ , asymptotically the same behavior as (1) and the differential equation (2) has a Lipschitz continuous right hand side, and has therefore a well defined solution.

The evolution of the output inventory position  $q$  is modeled in the same way, where the influx into the output inventory is the time delayed outflux of the input inventory, i.e.  $f^{in}(t) \rightarrow g(t - \tau)$  holds, with  $\tau$  the time it takes the supplier to process an item. The capacity  $\mu$  is replaced by the demand  $w$ , i.e. the supplier cannot ship at a rate greater than the current demand from the output inventory. Therefore, the output inventory position  $q$  will satisfy  $\frac{dq}{dt} = g(t - \tau) - \min\{w, \frac{q}{\varepsilon}\}$ . So, in summary, the dynamics of each supplier  $S_j, j = 1 : J$ , are given by

$$(a) p'_j(t) = f_j^{in}(t) - g_j(t), \quad (b) g_j(t) = \min\{\mu_j, \frac{p_j}{\varepsilon}\}, \quad (3)$$

$$(c) q'_j(t) = g_j(t - \tau_j) - f_j(t), \quad (d) f_j(t) = \min\{w_j, \frac{q_j}{\varepsilon}\}. \quad (4)$$

For notational convenience we include external suppliers of raw materials and final customers as suppliers in the network by formally defining raw material suppliers as suppliers with an infinite output inventory and customers as suppliers with zero production capacity.

It remains to connect the dynamics of the individual suppliers through the fluxes  $f_j$  and  $f_j^{in}$  in (3). This is done by Kirchhoff's law. We define a  $J \times J$  distribution matrix  $A$  with nonnegative entries  $A_{jk}$ , and set

$$f_j^{in} = \sum_{k=1}^J A_{jk} f_k, \quad j = 1 : J \quad (5)$$

So the product flux  $\Phi_{jk}$  from  $S_k$  to  $S_j$  is given by  $\Phi_{jk} = A_{jk} f_k$ , and  $A_{jk}$  is the percentage of the output of supplier  $k$  sent to supplier  $j$ . We denote the column sums of the matrix  $A$  by  $a$  and by  $a_k$  the percentage of product shipped back into the system by supplier  $S_k$ ; hence  $a_k = \sum_{j=1}^J A_{jk}$ ,  $a^T = \mathbf{1}^T A$ . If there is no loss of product during shipping (which assumed to be instantaneous), then the column sums  $a_k$  will equal unity, except for those nodes  $S_k$  corresponding to final customers.

The demand  $w_j$  on supplier  $S_j$  in (3)(d) is given by the orders placed. The modeling is analogously as for the distribution rates. Assuming that the supplier  $S_k$  places orders to supplier  $S_j$  at a rate  $\Omega_{jk}$ , i.e.,

$$d_j = \sum_{k=1}^J \Omega_{jk}, \quad j = 1 : J, \quad w = \Omega \mathbf{1} \quad (6)$$

The matrix  $\Omega$  defines the topology of the network, since each supplier will only place orders to a limited number of other suppliers. The elements of the matrices  $A$  and  $\Omega$  are the distribution and order policy decision variables. However, the distribution policy cannot be chosen independently of the order policy. Not only can supplier  $S_k$  not ship more from the output inventory than the total demand  $w_k$  (as enforced by the form of the outflux function  $f$  in (3)(d)). He also cannot ship to the individual supplier  $S_j$  at a rate greater than  $S_j$  is ordering. That is, we have the additional constraints

$$\Phi_{jk} = A_{jk} f_k \leq \Omega_{kj}, \quad j, k = 1 : J \quad (7)$$

The constraint (7) is a dynamic and nonlinear constraint, since all terms in (7) depend on time and the fluxes  $f_k$  are given by (3). The constraint can be satisfied by choosing an a priori rule for the distribution policy (which is not dependent on the dynamics):  $A_{jk} = \frac{1}{w_k} \Omega_{kj}$ . This reduces (7) to  $f_k \leq w_k$  which is enforced by (3) automatically. This choice can be understood as follows.

1. If the output inventory is non-empty ( $q_k = O(1)$ ,  $f_k = d_k$ ): Satisfy all demands  $\Phi_{jk} = \Omega_{kj}$ .
2. If the output inventory is empty  $q_k < \varepsilon d_k$ ,  $f_k = \frac{q_k}{\varepsilon} \approx g_k(t - \tau_k) < d_k$ : Distribute the output proportionally to the demand. Set  $\Phi_{jk} = A_{jk} f_k = \frac{f_k}{d_k} \Omega_{kj}$ .

Given a certain order policy, any other distribution policy will have to use information about the current state of the system (the values of  $f_k(t)$  and  $d_k(t)$ ) to enforce the constraint (7).

*Remark 1.* Up to this point we assumed that, if supplier  $S_j$  cannot satisfy all orders at any point in time, these orders are lost. A more realistic model allows for the orders to be filled at a later point in time (occurring a penalty which has to be included in a cost functional for optimization). We remove the constraint that the output inventory position  $q_j(t)$  in (3) remains non-negative, and define the inventory position as  $q_j$  for positive  $q_j$  and the backlog as  $-q_j$  for negative  $q_j$ . In this case (3) (c)(d) have to be replaced by

$$q'_j(t) = g_j(t - \tau_j) - w_j(t), \quad f_j = H(q_j)w_j$$

for  $H$  being the Heavi-side function.

## 2.1 The money flow

One of the purposes of the model developed so far is the study of the evolution of bankruptcies in a given network. To this end, it is necessary to include cash flow into the model. Money flows in the opposite direction of product in the network, and is weighted by a price. We denote by  $r_j$  the price per product unit supplier  $S_j$  charges to deliver. Therefore, the flow of money  $\psi_{jk}$  from supplier  $S_k$  to supplier  $S_j$  is given by  $\psi_{jk} = \Phi_{kj}r_j$ . Furthermore, we assume that each supplier  $S_j$  has certain production costs for delivering the product. The production costs per product unit supplier  $S_j$  are denoted by  $\mathbf{r}_j$ . Hence, the money supply  $\sigma_j$  of supplier  $S_j$  evolves according to

$$\sigma'_j(t) = \sum_{k=1}^J \psi_{jk} - \psi_{kj} - \sum_{k=1}^J \mathbf{r}_k f_k = a_j f_j r_j - \sum_{k=1}^J A_{jk} f_k r_k - \sum_{k=1}^J \mathbf{r}_k f_k, \quad (8)$$

If we assume that no credit is extended, i.e. the supplier  $S_j$  has to stop ordering once its money supply is exhausted, we obtain the condition

$$\Omega_{kj} = 0 \text{ if } \sigma_j = 0 \quad (9)$$

The above model allows for the simulation of the flow of product and payments on arbitrary complex networks, given a certain order and pricing policy, i.e. once the order rates  $\Omega_{jk}(t)$  and the prices  $r_j(t)$  are chosen. Choosing optimal policies by solving a constrained optimization problem is subject of Section 3.

## 3 Computing optimal order and distribution policies

We are interested in an optimal choice of the given order policies  $\Omega_{jk}$  and distribution policies  $A_{jk}$ . We consider the choice to be optimal, if the profit of the supply chain  $\sum_{j=1}^J \sigma_j(T)$  at some final time  $T$  is maximal. Constraints to this optimization problem are the dynamics of the supply chain as well as constraints on order flows, production capacities, inventories and possible bankruptcy.

This is described by the following maximization problem for  $\Omega_{jk}$  and  $A_{jk}$ .

$$\max_{\Omega_{kj}, A_{jk}} \sum_{j=1}^J \sigma_j(T) \text{ subject to (3), (5), (6), (7), (8), (9).} \quad (10)$$

Numerically, this problem is solved using a mixed-integer programming formulation as studied e.g. in [4, 6]. We study some analytical properties of the maximization problem.

**Lemma 1.** *As long as there is no bankruptcy, the internal pricing has no influence on the order  $\Omega_{kj}$  and distribution policy  $A_{jk}$ .*

In fact, due to equation (5), we have  $\sum_j A_{jk} = 1$  and hence  $A_{jk} f_k = 0 \forall k$ , for customers  $j$ , and  $A_{jk} f_j = 0 \forall j$ , for raw material suppliers  $k$ , respectively. We introduce the index sets  $\mathcal{A}_C, \mathcal{A}_M$  and  $\mathcal{A}_S$  for customers, raw material suppliers and the remaining suppliers. Then, we obtain

$$\sum_{j,k} A_{kj} f_j r_j - A_{jk} f_k r_k = \sum_{j \in \mathcal{A}_C} f_j r_j - \sum_{j \in \mathcal{A}_M} f_j r_j.$$

For an initial money supply of  $\sigma_j(t=0) = 0$  we obtain by integrating (8)

$$\sum_j \sigma_j(T) = \int_0^T \left( \sum_j -r_j f_j + \sum_{j \in \mathcal{A}_C} f_j r_j - \sum_{j \in \mathcal{A}_M} f_j r_j \right) dt. \quad (11)$$

Hence, only the production costs but *not* the internal pricing effect the cost functional (as long as there is no bankruptcy).

Second, we reformulate equations (3) in order to obtain a partial differential equation. This approach is analogously to the presentation in ([6]): We assume each supplier as unit length. The delay in  $\tau_j$  is then modeled by a function  $\rho(x, t)$  satisfying

$$\begin{aligned} p'(t) &= f^{in} - \rho(0, t), \quad q'(t) = \rho(1, t) - f(t), \\ \rho_t + \frac{1}{\tau} g_x &= 0, \quad \rho(0, t) = \min\{\mu, \frac{p}{\epsilon}\}, \quad f = \min\{w, \frac{q}{\epsilon}\}, \\ g(t) &= \rho(0, t). \end{aligned}$$

The latter set of equations is in fact an Upwind discretization of a partial differential equation for the part densities  $\rho$  given by

$$\partial_t \rho + \partial_x \min\{\nu, v\rho\} = 0, \quad g(0, t) = f^{in} \quad (12)$$

and where  $\nu(x) = \mu \chi_{x \leq \frac{1}{2}} + w \chi_{x \geq \frac{1}{2}}$ ,  $v = \frac{1}{\tau}$  and  $p = \epsilon \rho(0-, t)$  and  $q = \epsilon \rho(1+, t)$ . This equation has to be solved for  $\rho_j$ , i.e., the part density for supplier  $S_j$ .

Third, if we denote by  $\psi_j := \min\{\nu_j, v_j \rho_j\}$ , we obtain  $f_j = \psi_j(1, t)$  and  $f_j^{in} = \psi_j(0, t)$ . Assuming an initially empty supply chain  $\rho_j(x, 0) = 0$  for all

suppliers  $j$ , and equal prices per product  $r_j \equiv r$ , we obtain from (12) upon integrating

$$\begin{aligned}
\sum_{j=1}^J \sigma_j(T) &= - \int_0^T \sum_{j=1}^J \mathbf{r}_j f_j dt + \int_0^T \sum_{j \in \mathcal{A}_C} f_j r - \sum_{j \in \mathcal{A}_M} f_j r dt \\
&= - \int_0^T \sum_{j=1}^J \mathbf{r}_j \psi_j(1, t) dt + \int_0^T r \left( \sum_{j \in \mathcal{A}_C} \psi_j(1, t) - \sum_{j \in \mathcal{A}_M} \psi_j(1, t) \right) dt \\
&= - \int_0^T \sum_{j=1}^J \mathbf{r}_j \psi_j(1, t) dt + \sum_{j=1}^J \int_0^T r (\psi_j(0, t) - \psi_j(1, t)) dt \\
&= - \int_0^T \sum_{j=1}^J \mathbf{r}_j \psi_j(1, t) dt - r \sum_{j=1}^J \int_0^1 \rho_j(x, T) dx.
\end{aligned}$$

Summarizing, we proved:

**Lemma 2.** *If  $\mathbf{r}_j \equiv 0$ , then maximizing the costs at time  $T$  using cost functional (11), is equivalent to minimizing the load in the complete network at time  $t = T$  where the load is the number of parts  $\int \rho_j dx$  in supplier  $S_j$ .*

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