

## EXERCISES

1.  $y = (x^4 - 3x^2 + 5)^3 \Rightarrow$   
 $y' = 3(x^4 - 3x^2 + 5)^2 \frac{d}{dx}(x^4 - 3x^2 + 5) = 3(x^4 - 3x^2 + 5)^2(4x^3 - 6x) = 6x(x^4 - 3x^2 + 5)^2(2x^2 - 3)$
2.  $y = \cos(\tan x) \Rightarrow y' = -\sin(\tan x) \frac{d}{dx}(\tan x) = -\sin(\tan x)(\sec^2 x)$
3.  $y = \sqrt{x} + \frac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \Rightarrow y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \frac{1}{2\sqrt{x}} - \frac{4}{3\sqrt[3]{x^7}}$
4.  $y = \frac{3x - 2}{\sqrt{2x + 1}} \Rightarrow$   
 $y' = \frac{\sqrt{2x + 1}(3) - (3x - 2)\frac{1}{2}(2x + 1)^{-1/2}(2)}{(\sqrt{2x + 1})^2} \cdot \frac{(2x + 1)^{1/2}}{(2x + 1)^{1/2}} = \frac{3(2x + 1) - (3x - 2)}{(2x + 1)^{3/2}} = \frac{3x + 5}{(2x + 1)^{3/2}}$
5.  $y = 2x\sqrt{x^2 + 1} \Rightarrow$   
 $y' = 2x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x) + \sqrt{x^2 + 1}(2) = \frac{2x^2}{\sqrt{x^2 + 1}} + 2\sqrt{x^2 + 1} = \frac{2x^2 + 2(x^2 + 1)}{\sqrt{x^2 + 1}} = \frac{2(2x^2 + 1)}{\sqrt{x^2 + 1}}$
6.  $y = \frac{e^x}{1 + x^2} \Rightarrow y' = \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(x^2 - 2x + 1)}{(1 + x^2)^2} = \frac{e^x(x - 1)^2}{(1 + x^2)^2}$
7.  $y = e^{\sin 2\theta} \Rightarrow y' = e^{\sin 2\theta} \frac{d}{d\theta}(\sin 2\theta) = e^{\sin 2\theta}(\cos 2\theta)(2) = 2 \cos 2\theta e^{\sin 2\theta}$
8.  $y = e^{-t}(t^2 - 2t + 2) \Rightarrow$   
 $y' = e^{-t}(2t - 2) + (t^2 - 2t + 2)(-e^{-t}) = e^{-t}(2t - 2 - t^2 + 2t - 2) = e^{-t}(-t^2 + 4t - 4)$
9.  $y = \frac{t}{1 - t^2} \Rightarrow y' = \frac{(1 - t^2)(1) - t(-2t)}{(1 - t^2)^2} = \frac{1 - t^2 + 2t^2}{(1 - t^2)^2} = \frac{t^2 + 1}{(1 - t^2)^2}$
10.  $y = \sin^{-1}(e^x) \Rightarrow y' = \frac{1}{\sqrt{1 - (e^x)^2}} \cdot e^x = e^x / \sqrt{1 - e^{2x}}$
11.  $y = xe^{-1/x} \Rightarrow y' = xe^{-1/x}(1/x^2) + e^{-1/x} \cdot 1 = e^{-1/x}(1/x + 1)$
12.  $y = x^r e^{sx} \Rightarrow y' = x^r(se^{sx}) + e^{sx}(rx^{r-1}) = e^{sx}x^{r-1}(sx + r)$
13.  $y = \tan\sqrt{1 - x} \Rightarrow y' = (\sec^2\sqrt{1 - x})\left(\frac{1}{2\sqrt{1 - x}}\right)(-1) = -\frac{\sec^2\sqrt{1 - x}}{2\sqrt{1 - x}}$
14. Using the Reciprocal Rule,  $g(x) = \frac{1}{f(x)} \Rightarrow g'(x) = -\frac{f'(x)}{[f(x)]^2}$ , we have  $y = \frac{1}{\sin(x - \sin x)} \Rightarrow$   
 $y' = -\frac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}$
15.  $\frac{d}{dx}(xy^4 + x^2y) = \frac{d}{dx}(x + 3y) \Rightarrow x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow$   
 $y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \Rightarrow y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$
16.  $y = \ln(\csc 5x) \Rightarrow y' = \frac{1}{\csc 5x}(-\csc 5x \cot 5x)(5) = -5 \cot 5x$

17.  $y = \frac{\sec 2\theta}{1 + \tan 2\theta} \Rightarrow$   

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta [(1 + \tan 2\theta) \tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2 \sec 2\theta (\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \quad [1 + \tan^2 x = \sec^2 x]$$
18.  $\frac{d}{dx} (x^2 \cos y + \sin 2y) = \frac{d}{dx} (xy) \Rightarrow x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \Rightarrow$   

$$y'(-x^2 \sin y + 2 \cos 2y - x) = y - 2x \cos y \Rightarrow y' = \frac{y - 2x \cos y}{2 \cos 2y - x^2 \sin y - x}$$
19.  $y = e^{cx}(c \sin x - \cos x) \Rightarrow$   

$$y' = e^{cx}(c \cos x + \sin x) + ce^{cx}(c \sin x - \cos x)$$

$$= e^{cx}(c^2 \sin x - c \cos x + c \cos x + \sin x) = e^{cx}(c^2 \sin x + \sin x) = e^{cx} \sin x (c^2 + 1)$$
20.  $y = \ln(x^2 e^x) = \ln x^2 + \ln e^x = 2 \ln x + x \Rightarrow y' = 2/x + 1$
21.  $y = e^{e^x} \Rightarrow y' = e^{e^x} \frac{d}{dx} (e^x) = e^{e^x} e^x = e^{x+e^x}$
22.  $y = \sec(1 + x^2) \Rightarrow y' = 2x \sec(1 + x^2) \tan(1 + x^2)$
23.  $y = (1 - x^{-1})^{-1} \Rightarrow$   

$$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x - 1)/x)^{-2}x^{-2} = -(x - 1)^{-2}$$
24.  $y = (x + \sqrt{x})^{-1/3} \Rightarrow y' = -\frac{1}{3}(x + \sqrt{x})^{-4/3} \left(1 + \frac{1}{2\sqrt{x}}\right)$
25.  $\sin(xy) = x^2 - y \Rightarrow \cos(xy)(xy' + y \cdot 1) = 2x - y' \Rightarrow x \cos(xy)y' + y' = 2x - y \cos(xy) \Rightarrow$   

$$y'[x \cos(xy) + 1] = 2x - y \cos(xy) \Rightarrow y' = \frac{2x - y \cos(xy)}{x \cos(xy) + 1}$$
26.  $y = \sqrt{\sin \sqrt{x}} \Rightarrow y' = \frac{1}{2}(\sin \sqrt{x})^{-1/2}(\cos \sqrt{x}) \left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos \sqrt{x}}{4\sqrt{x} \sin \sqrt{x}}$
27.  $y = \log_5(1 + 2x) \Rightarrow y' = \frac{1}{(1 + 2x) \ln 5} \frac{d}{dx} (1 + 2x) = \frac{2}{(1 + 2x) \ln 5}$
28.  $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x = x \ln \cos x \Rightarrow \frac{y'}{y} = x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \Rightarrow$   

$$y' = (\cos x)^x (\ln \cos x - x \tan x)$$
29.  $y = \ln \sin x - \frac{1}{2} \sin^2 x \Rightarrow y' = \frac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \cot x - \sin x \cos x$

$$30. y = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} \Rightarrow$$

$$\ln y = \ln \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} = \ln(x^2 + 1)^4 - \ln[(2x + 1)^3(3x - 1)^5]$$

$$= 4 \ln(x^2 + 1) - [\ln(2x + 1)^3 + \ln(3x - 1)^5] = 4 \ln(x^2 + 1) - 3 \ln(2x + 1) - 5 \ln(3x - 1) \Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2 + 1} \cdot 2x - 3 \cdot \frac{1}{2x + 1} \cdot 2 - 5 \cdot \frac{1}{3x - 1} \cdot 3 \Rightarrow$$

$$y' = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5} \left( \frac{8x}{x^2 + 1} - \frac{6}{2x + 1} - \frac{15}{3x - 1} \right).$$

[The answer could be simplified to  $y' = -\frac{(x^2 + 56x + 9)(x^2 + 1)^3}{(2x + 1)^4(3x - 1)^6}$ , but this is unnecessary.]

$$31. y = x \tan^{-1}(4x) \Rightarrow y' = x \cdot \frac{1}{1 + (4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \frac{4x}{1 + 16x^2} + \tan^{-1}(4x)$$

$$32. y = e^{\cos x} + \cos(e^x) \Rightarrow y' = e^{\cos x}(-\sin x) + [-\sin(e^x) \cdot e^x] = -\sin x e^{\cos x} - e^x \sin(e^x)$$

$$33. y = \ln |\sec 5x + \tan 5x| \Rightarrow$$

$$y' = \frac{1}{\sec 5x + \tan 5x} (\sec 5x \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \frac{5 \sec 5x (\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5 \sec 5x$$

$$34. y = 10^{\tan \pi \theta} \Rightarrow y' = 10^{\tan \pi \theta} \cdot \ln 10 \cdot \sec^2 \pi \theta \cdot \pi = \pi (\ln 10) 10^{\tan \pi \theta} \sec^2 \pi \theta$$

$$35. y = \cot(3x^2 + 5) \Rightarrow y' = -\csc^2(3x^2 + 5)(6x) = -6x \csc^2(3x^2 + 5)$$

$$36. y = \sqrt{t \ln(t^4)} \Rightarrow$$

$$y' = \frac{1}{2} [t \ln(t^4)]^{-1/2} \frac{d}{dt} [t \ln(t^4)] = \frac{1}{2 \sqrt{t \ln(t^4)}} \cdot \left[ 1 \cdot \ln(t^4) + t \cdot \frac{1}{t^4} \cdot 4t^3 \right]$$

$$= \frac{1}{2 \sqrt{t \ln(t^4)}} \cdot [\ln(t^4) + 4] = \frac{\ln(t^4) + 4}{2 \sqrt{t \ln(t^4)}}$$

Or: Since  $y$  is only defined for  $t > 0$ , we can write  $y = \sqrt{t \cdot 4 \ln t} = 2 \sqrt{t \ln t}$ . Then

$$y' = 2 \cdot \frac{1}{2 \sqrt{t \ln t}} \cdot \left( 1 \cdot \ln t + t \cdot \frac{1}{t} \right) = \frac{\ln t + 1}{\sqrt{t \ln t}}. \text{ This agrees with our first answer since}$$

$$\frac{\ln(t^4) + 4}{2 \sqrt{t \ln(t^4)}} = \frac{4 \ln t + 4}{2 \sqrt{t \cdot 4 \ln t}} = \frac{4(\ln t + 1)}{2 \cdot 2 \sqrt{t \ln t}} = \frac{\ln t + 1}{\sqrt{t \ln t}}.$$

$$37. y = \sin(\tan \sqrt{1 + x^3}) \Rightarrow y' = \cos(\tan \sqrt{1 + x^3}) (\sec^2 \sqrt{1 + x^3}) [3x^2 / (2\sqrt{1 + x^3})]$$

$$38. y = \arctan(\arcsin \sqrt{x}) \Rightarrow y' = \frac{1}{1 + (\arcsin \sqrt{x})^2} \cdot \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}}$$

$$39. y = \tan^2(\sin \theta) = [\tan(\sin \theta)]^2 \Rightarrow y' = 2[\tan(\sin \theta)] \cdot \sec^2(\sin \theta) \cdot \cos \theta$$

$$40. xe^y = y - 1 \Rightarrow xe^y y' + e^y = y' \Rightarrow e^y = y' - xe^y y' \Rightarrow y' = e^y / (1 - xe^y)$$

$$41. y = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7} \Rightarrow \ln y = \frac{1}{2} \ln(x+1) + 5 \ln(2-x) - 7 \ln(x+3) \Rightarrow$$

$$\frac{y'}{y} = \frac{1}{2(x+1)} + \frac{-5}{2-x} - \frac{7}{x+3} \Rightarrow y' = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7} \left[ \frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3} \right] \text{ or}$$

$$y' = \frac{(2-x)^4(3x^2 - 55x - 52)}{2\sqrt{x+1}(x+3)^8}.$$

$$42. y = \frac{(x + \lambda)^4}{x^4 + \lambda^4} \Rightarrow y' = \frac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \frac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$$

$$43. y = x \sinh(x^2) \Rightarrow y' = x \cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2 \cosh(x^2) + \sinh(x^2)$$

$$44. y = (\sin mx)/x \Rightarrow y' = (mx \cos mx - \sin mx)/x^2$$

$$45. y = \ln(\cosh 3x) \Rightarrow y' = (1/\cosh 3x)(\sinh 3x)(3) = 3 \tanh 3x$$

$$46. y = \ln \left| \frac{x^2 - 4}{2x + 5} \right| = \ln|x^2 - 4| - \ln|2x + 5| \Rightarrow y' = \frac{2x}{x^2 - 4} - \frac{2}{2x + 5} \text{ or } \frac{2(x + 1)(x + 4)}{(x + 2)(x - 2)(2x + 5)}$$

$$47. y = \cosh^{-1}(\sinh x) \Rightarrow y' = \frac{1}{\sqrt{(\sinh x)^2 - 1}} \cdot \cosh x = \frac{\cosh x}{\sqrt{\sinh^2 x - 1}}$$

$$48. y = x \tanh^{-1} \sqrt{x} \Rightarrow y' = \tanh^{-1} \sqrt{x} + x \frac{1}{1 - (\sqrt{x})^2} \frac{1}{2\sqrt{x}} = \tanh^{-1} \sqrt{x} + \frac{\sqrt{x}}{2(1 - x)}$$

$$49. f(t) = \sqrt{4t + 1} \Rightarrow f'(t) = \frac{1}{2}(4t + 1)^{-1/2} \cdot 4 = 2(4t + 1)^{-1/2} \Rightarrow \\ f''(t) = 2(-\frac{1}{2})(4t + 1)^{-3/2} \cdot 4 = -4/(4t + 1)^{3/2}, \text{ so } f''(2) = -4/9^{3/2} = -\frac{4}{27}.$$

$$50. g(\theta) = \theta \sin \theta \Rightarrow g'(\theta) = \theta \cos \theta + \sin \theta \cdot 1 \Rightarrow g''(\theta) = \theta(-\sin \theta) + \cos \theta \cdot 1 + \cos \theta = 2 \cos \theta - \theta \sin \theta, \\ \text{so } g''(\pi/6) = 2 \cos(\pi/6) - (\pi/6) \sin(\pi/6) = 2(\sqrt{3}/2) - (\pi/6)(1/2) = \sqrt{3} - \pi/12.$$

$$51. x^6 + y^6 = 1 \Rightarrow 6x^5 + 6y^5 y' = 0 \Rightarrow y' = -x^5/y^5 \Rightarrow \\ y'' = -\frac{y^5(5x^4) - x^5(5y^4 y')}{(y^5)^2} = -\frac{5x^4 y^4 [y - x(-x^5/y^5)]}{y^{10}} = -\frac{5x^4 [(y^6 + x^6)/y^5]}{y^6} = -\frac{5x^4}{y^{11}}$$

$$52. f(x) = (2 - x)^{-1} \Rightarrow f'(x) = (2 - x)^{-2} \Rightarrow f''(x) = 2(2 - x)^{-3} \Rightarrow f'''(x) = 2 \cdot 3(2 - x)^{-4} \Rightarrow \\ f^{(4)}(x) = 2 \cdot 3 \cdot 4(2 - x)^{-5}. \text{ In general, } f^{(n)}(x) = 2 \cdot 3 \cdot 4 \cdots n(2 - x)^{-(n+1)} = \frac{n!}{(2 - x)^{(n+1)}}.$$

53. We first show it is true for  $n = 1$ :  $f(x) = xe^x \Rightarrow f'(x) = xe^x + e^x = (x + 1)e^x$ . We now assume it is true for  $n = k$ :  $f^{(k)}(x) = (x + k)e^x$ . With this assumption, we must show it is true for  $n = k + 1$ :

$$f^{(k+1)}(x) = \frac{d}{dx} [f^{(k)}(x)] = \frac{d}{dx} [(x + k)e^x] = (x + k)e^x + e^x = [(x + k) + 1]e^x = [x + (k + 1)]e^x.$$

Therefore,  $f^{(n)}(x) = (x + n)e^x$  by mathematical induction.

$$54. \lim_{t \rightarrow 0} \frac{t^8}{\tan^3 2t} = \lim_{t \rightarrow 0} \frac{t^3 \cos^3 2t}{\sin^3 2t} = \lim_{t \rightarrow 0} \cos^3 2t \cdot \frac{1}{8 \frac{\sin^3 2t}{(2t)^3}} = \lim_{t \rightarrow 0} \frac{\cos^3 2t}{8 \left( \lim_{t \rightarrow 0} \frac{\sin 2t}{2t} \right)^3} = \frac{1}{8 \cdot 1^3} = \frac{1}{8}$$

$$55. y = 4 \sin^2 x \Rightarrow y' = 4 \cdot 2 \sin x \cos x. \text{ At } (\frac{\pi}{6}, 1), y' = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}, \text{ so an equation of the tangent line is } \\ y - 1 = 2\sqrt{3}(x - \frac{\pi}{6}), \text{ or } y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3.$$

$$56. y = \frac{x^2 - 1}{x^2 + 1} \Rightarrow y' = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}. \text{ At } (0, -1), y' = 0, \text{ so an equation of the } \\ \text{tangent line is } y + 1 = 0(x - 0), \text{ or } y = -1.$$

$$57. y = \sqrt{1 + 4 \sin x} \Rightarrow y' = \frac{1}{2}(1 + 4 \sin x)^{-1/2} \cdot 4 \cos x = \frac{2 \cos x}{\sqrt{1 + 4 \sin x}}. \text{ At } (0, 1), y' = \frac{2}{\sqrt{1}} = 2, \text{ so an } \\ \text{equation of the tangent line is } y - 1 = 2(x - 0), \text{ or } y = 2x + 1.$$