MAT 119P

Fall Semester 2000

The Simplex Method for General Linear Programs, October 12, 2000

This document replaces Section 4.4 of the textbook.

As I have said in class, Chapter 4, especially Section 4.4, has been badly written. It is confusing and misleading. Hence, this particular document.

Section 4.4 deals with solving general linear programs (abbr: LP). To solve a general LP, first some "cosmetic" changes need to be made. If, after these changes, the origin (i.e. the point where $x_i = 0$, for all i) is not a feasible point, an auxiliary LP must first be solved. The solution of this auxiliary LP provides a starting point for the simplex method in the original LP. At this point, the original LP is solved. More detailed instructions follow.

Solving a General LP

(1) Cosmetic changes (do if necessary)

- (a) Convert min to max.
- (b) Convert $\geq$ inequalities to $\leq$ inequalities.

(2) Introduce slack variables and write the LP in tableau form.

(3) If all of the Right-Hand Sides (RHS) are nonnegative (greater than or equal to zero), keep this tableau and proceed to (4). Otherwise, do the following:

- (a) Between the Basic Variable (BV) column and the column for the first variable, insert a column consisting entirely of -1's; this will be called the *x_0 column*.
- (b) Replace the last row with $[1|1\ 0\ 0\ \cdots\ 0|0]$. The resulting tableau represents the *auxiliary LP*.
- (c) Put a pivot in the x_0 column, in the row which has the smallest (most negative) RHS value.
- (d) Proceed with the simplex method on this LP.
- (e) If $x_0 > 0$ for the optimal solution, the original problem is *infeasible*; there are no solutions, because no point satisfies all of the inequalities. Stop here.
- (f) Otherwise, form a new tableau the following way: Remove the x_0 column and replace the last row with the last row of the original LP.
- (g) The pivots in the solution to the auxiliary LP are probably not pivots in the tableau you formed in (f). Fix that now: add/subtract a multiple of each row to/from the last row.

(4) Solve the LP as usual; apply the simplex method.

(5) Extract the solution (if there is one) from the tableau. If the original problem was a minimization problem (see (1a)), you will have to multiply the objective value by -1, but leave the values of the x_i's alone. *Always solve the problem that you were asked to solve.*

An Example

For example, consider the following LP (which I did in class): Minimize the quantity $z = -3x_1 - x_2$ subject to the following inequalities:

$$\begin{array}{rcrcr} x_1 & - & x_2 & \leq & -1 \\ x_1 & + & x_2 & \geq & 3 \\ 2x_1 & + & x_2 & \leq & 4 \end{array}$$

$$x_1, x_2 \geq 0$$

We start with Step (1a). Since we are asked to minimize z, we will instead maximize $z' = -z = 3x_1 + x_2$ and convert this back to the solution we want at the end of the problem.

Step (1b) requires that we convert $\geq$ inequalities to $\leq$ inequalities. The first and third inequalities are already $\leq$ inequalities, but the second one isn't. To fix this, we multiply both sides of the second inequality by -1, which also reverses the direction of the $\geq$ symbol. Hence we are now maximizing the quantity $z' = 3x_1 + x_2$ subject to the following constraints:

$$\begin{array}{rcrcr} x_1 & - & x_2 & \leq & -1 \\ -x_1 & - & x_2 & \leq & -3 \\ 2x_1 & + & x_2 & \leq & 4 \end{array}$$

$$x_1, x_2 \geq 0$$

Step (2) requires that we introduce slack variables and put this LP into tableau form. The slack variables will be called x_3, x_4, and x_5. The tableau looks like:

$$\begin{array}{c} \text{BV} \\ x_3 \\ x_4 \\ x_5 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} z' & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \end{array} \\ \left[\begin{array}{c|rrrrr|r} 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 & 0 & -3 \\ 0 & 2 & 1 & 0 & 0 & 1 & 4 \\ \hline 1 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Now Step (3) says to examine the RHS's of this tableau; they are -1, -3, and 4. Since -1 is negative, the solution we want to start with ($x_1 = x_2 = 0$, $x_3 = -1$, $x_4 = -3$, $x_5 = 4$) is infeasible, because $x_3 < 0$. Hence, we need a different starting point and thus proceed through Steps (3a)–(3g).

Step (3a) says to introduce a new variable (x_0), inserting a column of -1's between the first two columns of our tableau:

$$\begin{array}{c} \text{BV} \\ x_3 \\ x_4 \\ x_5 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} z' & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \end{array} \\ \left[\begin{array}{c|rrrrrr|r} 0 & -1 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 & -3 \\ 0 & -1 & 2 & 1 & 0 & 0 & 1 & 4 \\ \hline 1 & -1 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Step (3b) says to replace the final row of our tableau with $[1|1\ 0\ 0\ \cdots\ 0|0]$. This gives us our auxiliary LP:

BV	z'	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	-1	1	-1	1	0	0	-1
x_4	0	-1	-1	-1	0	1	0	-3
x_5	0	-1	2	1	0	0	1	4
	1	1	0	0	0	0	0	0

Notice that the "point" which this tableau represents ($x_0 = x_1 = x_2 = 0$, $x_3 = -1$, $x_4 = -3$, $x_5 = 4$) is infeasible; Step (3c) fixes this. The entry in the x_0 column and the x_4 row will now be turned into a pivot. First, we multiply that row by -1:

BV	z'	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	-1	1	-1	1	0	0	-1
(x_4)	0	$\boxed{1}$	1	1	0	-1	0	3
x_5	0	-1	2	1	0	0	1	4
	1	1	0	0	0	0	0	0

Now we add Row 2 to Rows and 3, and we subtract Row 2 from Row 4; this yields

BV	z'	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	2	0	1	-1	0	2
x_0	0	1	1	1	0	-1	0	3
x_5	0	0	3	2	0	-1	1	7
	1	0	-1	-1	0	1	0	-3

which is a feasible tableau; the point this tableau represents is $x_1 = x_2 = x_4 = 0$, $x_3 = 2$, $x_0 = 3$, $x_5 = 7$. Note that the entry in the last row and the last column (-3) is negative. This number will increase, possibly getting as high as 0, but it should *never* become strictly positive. (If a positive number shows up in this position, we made a mistake somewhere.)

Step (3d) says to proceed with the simplex method on this tableau: Find a negative entry in the bottom row (other than the RHS), look at the positive numbers above it, find the smallest ratio, etc. It turns out that after one pivot, we reach an optimum solution; the final tableau turns out to be

BV	z'	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_1	0	0	1	0	$1/2$	$-1/2$	0	1
x_2	0	1	0	1	$-1/2$	$-1/2$	0	2
x_5	0	-2	0	0	$-1/2$	$3/2$	1	0
	1	1	0	0	0	0	0	0

This solution has $x_0 = x_3 = x_4 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_5 = 0$. In particular, $x_0 = 0$, and Step (3e) tells us that the rest of this solution ($x_3 = x_4 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_5 = 0$) is a feasible point for the original LP. (If x_0 had been positive, we would have stopped here, because the original problem would have been infeasible.) Now we need to start the simplex method from this point.

Following Step (3f), we remove the x_0 column from the solution. We now have the tableau

$$\begin{array}{c} \text{BV} \\ x_1 \\ x_2 \\ x_5 \\ \\ \end{array} \begin{array}{c|cccccc|c} z' & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\ 0 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

Step (3f) says to replace the bottom row of this tableau with the bottom row of the original tableau, to get

$$\begin{array}{c} \text{BV} \\ x_1 \\ x_2 \\ x_5 \\ \\ \end{array} \begin{array}{c|cccccc|c} z' & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\ 0 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline 1 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}.$$

We want pivots in the x_1, x_2, and x_5 columns. We don't quite have them; the x_1 column has a -3 in the last row, which keeps the 1 in that column from being a pivot. Step (3g) says to put a zero in this position. We do so by adding 3 times Row 1 to Row 4. To clear out the -1 in the x_2 column, we add Row 2 to Row 4. After doing this, we get:

$$\begin{array}{c} \text{BV} \\ x_1 \\ x_2 \\ x_5 \\ \\ \end{array} \begin{array}{c|cccccc|c} z' & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 & 2 \\ 0 & 0 & 0 & -1/2 & 3/2 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & -2 & 0 & 5 \end{array}$$

This is where we come to Step (4). We use the simplex method to solve the LP, starting from the above tableau. It turns out that after putting a pivot in the x_4 column, we reach an optimum solution:

$$\begin{array}{c} \text{BV} \\ x_1 \\ x_2 \\ x_4 \\ \\ \end{array} \begin{array}{c|cccccc|c} z' & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ 0 & 1 & 0 & 1/3 & 0 & 1/3 & 1 \\ 0 & 0 & 1 & -2/3 & 0 & 1/3 & 2 \\ 0 & 0 & 0 & -1/3 & 1 & 2/3 & 0 \\ \hline 1 & 0 & 0 & 1/3 & 0 & 4/3 & 5 \end{array}$$

We now finish things off with Step (5). This solution has $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$, and $z' = 5$. Recall that we wanted to minimize $z = -3x_1 - x_2$. The solution we have that maximizes z' also minimizes z; we can plug in $x_1 = 1$ and $x_2 = 2$ to get our value of z, or we note that $z = -z' = -5$.