10683. *Proposed by Harry Tamvakis, University of Pennsylvania, Philadelphia, PA.* Let $a_1, \ldots, a_n$ be a sequence of nonzero real numbers, exactly p of which are positive. Characterize the pairs (n, p) such that exactly half of the possible products $a_i a_j a_k$ with $i < j < k$ are positive.

Solution by Christopher Carl Heckman, Georgia Institute of Technology: **RECONSTRUCTED** This isn't too difficult. Define the numbers x_i by

$$x_i = \begin{cases} +1, & \text{if } a_i > 0; \\ -1, & \text{if } a_i < 0. \end{cases}$$

Note that $x_i x_j x_k = +1$ when $a_i a_j a_k > 0$, and $x_i x_j x_k = -1$ when $a_i a_j a_k < 0$. Notice that $\sum_i x_i^3 = \sum_i x_i = 1 \cdot p + (-1) \cdot (n-p) = 2p - n$, and $\sum_i x_i^2 = n$, so

$$\begin{aligned} (2p-n)^3 = (x_1 + \cdots + x_3)^3 &= \sum_{i,j,k \text{ distinct}} x_i x_j x_k + 3 \sum_i \sum_{j \neq i} x_i^2 x_j + \sum_i x_i^3 \\ &= 3! \sum_{i<j<k} x_i x_j x_k + 3 \sum_i \sum_{j \neq i} x_i^2 x_j + \sum_i x_i = 6 \sum_{i<j<k} x_i x_j x_k + 3 \sum_j \sum_{i \neq j} x_j + \sum_i x_i \\ &= 6 \sum_{i<j<k} x_i x_j x_k + 3 \left(\sum_j x_j \right) \left(\sum_{i \neq j} 1 \right) + \sum_i x_i \\ &= 6 \cdot 0 + 3(2p-n)(n-1) + (2p-n) = (2p-n)(3n-2). \end{aligned}$$

Note that we had $\sum x_i x_j x_k = 0$ as a consequence of the assumption that exactly half of the possible products $a_i a_j a_k$ with $i < j < k$ are positive; the rest must be negative, as $x_i \neq 0$, and so the sum must be 0.

We now solve this equality for integral values of n and p. If $n = 2p$, then exactly half of the products $x_i x_j x_k$ are positive, and so exactly half of the products $a_i a_j a_k$ are positive (for $i < j < k$). Otherwise, $(2p-n)^2 = 3n - 2$, or $n^2 - (4p+3)n + (4p^2+2) = 0$. Applying the quadratic formula, we obtain

$$n = \frac{4p + 3 \pm \sqrt{(4p+3)^2 - 4(4p^2+2)}}{2} = 2p + \frac{3}{2} \pm \frac{1}{2}\sqrt{24p+1}.$$

For n to be rational, we need $24p + 1$ to be a perfect square; for n to be an integer, we also require that $\sqrt{24p+1}$ to be odd, and hence $24p + 1$ is an odd square. Thus we can write $24p + 1 = (2k+1)^2$, for some integer $k \geq 0$. Solving for p, we get $p = \frac{1}{6} k(k+1)$, which is an integer iff $k \equiv 0$ or $2 \pmod 3$. Substituting for p, we get

$$n = \frac{1}{3} k(k+1) + \frac{3}{2} \pm \frac{1}{2}(2k+1),$$

and each of these is a valid solution to the original problem, for any positive k; that is,

$$n \geq \frac{1}{3} k(k+1) + \frac{3}{2} - \frac{1}{2}(2k+1) \geq \frac{1}{6} k(k+1) = p,$$

for all integers k.[1] This exhausts all possibilities for (n, p). The characterization is thus: $n = 2p$; $0 \leq p \leq n < 3$,[2] or $(n, p) = \left(\frac{1}{3} k(k+1) + \frac{3}{2} \pm \frac{1}{2}(2k+1), \frac{1}{6} k(k+1) \right)$, for some integer $k \geq 0$ with $k \equiv 0$ or 2 (mod 3).[3]

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[1] In fact, the middle inequality is true for all real k except those strictly between 2 and 3.

[2] Vacuuously true.

[3] If $k = 0$, the result is vacuuously true.