10856. *Proposed by Andrei Jorza, "Moise Nicoara" High School, Arad, Romania.* Find all bounded convex polyhedra such that no three faces have the same number of edges.

Solution by Christopher Carl Heckman: A polyhedron $\mathcal{P}$ is a polytope of dimension three. Consider the graph H whose vertices are the corners of $\mathcal{P}$ and whose edges are the edges of $\mathcal{P}$. The graph H is planar (embedded in the obvious way) and 3-connected, and no three facial cycles have the same length. Let G be the planar dual of H; then G is a planar graph such that no three of its vertices have the same degree. We will now determine all possible graphs G.

Since $\mathcal{P}$ is a polytope, G is a simple graph, and hence every facial walk of G has length at least three. Using the Handshaking Lemma for H and Euler's formula for G ($v - e + f = 2$, where G has v vertices, e edges, and f faces), we have

$$2e = \sum_{\mathcal{F} \text{ a face of } G} |\mathcal{F}| \geq 3f = 3(2 + e - v) = 6 + 3e - 3v, \tag{1}$$

which implies that $3v \geq 6 + e$.

Now let v_i be the number of vertices of G of degree i; then

$$\sum_{i \geq 3} 6v_i = 6v \geq 12 + 2e = 12 + \sum_{i \geq 3} i v_i,$$

which also implies that

$$\sum_{i \geq 6} (6 - i)v_i + v_5 + 2v_4 + 3v_3 \geq 12. \tag{2}$$

Since $6 - i < 0$, for $i \geq 7$, $v_5 + 2v_4 + 3v_3 \geq 12$. Because G has at most two vertices of degree d, for all d, $v_5 + 2v_4 + 3v_3 \leq 2 + 4 + 6 = 12$. Hence, we must have $v_3 = v_4 = v_5 = 2$. This also implies that the sum in (2) must equal zero; hence G has no vertices of degree larger than six.

The degree sequence of G must be one of 5,5,4,4,3,3; 6,5,5,4,4,3,3; or 6,6,5,5,4,4,3,3.

Furthermore, G is a triangulation: Note that $v = 6 + v_6$ and $e = 12 + 3v_6$. Euler's formula implies that $f = 8 + 2v_6$. Consider (1) again. Since $2e = 6 + 3e - 3v$, the inequality must be an equality. Hence every face is bounded by a triangle.

If the degree sequence of G is 5,5,4,4,3,3, G is a unique graph. Since G has six vertices, the two vertices u_1 and u_2 of degree five are adjacent to each other. Then $G \setminus \{u_1, u_2\}$ has degree sequence 2,2,1,1 and can only be a path (on four vertices). This determines G uniquely. Furthermore, there is a polyhedron which yields the graph G: for example, the convex hull of the points $(0, \pm 1, 0)$, $(1, \pm 1, \pm 1)$ (representing four points), and $(2, 0, \pm 2)$.

A similar procedure can be used for the other two degree sequences. A polyhedron yielding a graph with a degree sequence of 6,5,5,4,4,3,3 (G is also unique) is the convex hull of the points $(0, \pm 1, 2)$, $(0, \pm 2, 0)$, $(\pm 2, \pm 1, 0)$ (again, representing four points), $(1, 2, 1)$, and $(-1, -2, 1)$.

There are two planar graphs with a degree sequence of 6,6,5,5,4,4,3,3, and they can be obtained from the polyhedra formed by the convex hull of the points $(0, 0, 0)$, $(1, 0, \pm 1)$, $(2, 1, \pm 2)$, $(3, 2, 0)$, $(3, 3, 0)$, $(2, 4, \pm 2)$, $(1, 5, \pm 1)$, and $(0, 5, 0)$; or the convex hull of the points $(0, \pm 2, 0)$, $(1, -2, 1)$, $(1, 2, -1)$, $(2, \pm 2, 0)$, $(2, 2, 2)$, $(2, -2, -2)$, $(3, -1, 3)$, $(3, 1, -3)$, and $(4, 0, \pm 4)$. □

This proof can be generalized to the following:

Theorem. *Let d be an integer which is at least two. Then there are a finite number of polyhedra (up to placement of vertices) such that no $d + 1$ faces have the same number of edges.*

Proof: Only changes from the above procedure are noted. The only fact that needs to be proved is that there are a finite number of possibilties for the resulting graph G. We start with the inequality (2).

Since there are at most d vertices of any degree,

$$6d \geq v_5 + 2v_4 + 3v_3 \geq 12 + \sum_{i \geq 6} (i - 6)v_i. \tag{3}$$

Now we will show that G has no vertices of degree at least $6d-5$. Suppose, to the contrary, that there is a vertex of degree D, where $D \geq 6d-5$. Then $D \geq 7$ as well, and $v_D \geq 1$. Substituting this into (3), we have:

$$6d - 12 \geq \sum_{i \geq 6} (i-6)v_i \geq (D-6)v_D \geq D - 6 \geq 6d - 11,$$

which is clearly a contradiction.

Hence, G contains at most d vertices with degree i, for all $3 \leq i \leq 6d-6$, and no vertices with any other degree; hence, G has at most $d(6d-8)$ vertices. Since the number of graphs with at most N vertices is finite for all N, the result follows. □