11196. *Proposed by Mohammad Hossein Mehrabi, Iran University of Science and Technology, Tehran, Iran.* Let A and B be real $n \times n$ matrices. Show that if $AB - BA$ is invertible and $A^2 + B^2 = \sqrt{3}(AB - BA)$, then n is a multiple of 6.

Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ: Let $M = A + iB$. Then, since A and B are real matrices,

$$\begin{aligned} M\overline{M} &= (A+iB)\overline{(A+iB)} = (A+iB)(A-iB) = (A^2+B^2) - i(AB-BA) \\ &= (A^2+B^2) - \frac{i(A^2+B^2)}{\sqrt{3}} = \left(1 - \frac{i}{\sqrt{3}}\right)(A^2+B^2). \end{aligned}$$

Now, let $\alpha = 1 - \dfrac{i}{\sqrt{3}}$ and take the determinant of both sides. Since $AB - BA$ is assumed to be invertible, $A^2 + B^2$ is also invertible (being a nonzero multiple of $AB - BA$), and

$$\det(M\overline{M}) = \det\big(\alpha(A^2+B^2)\big) = \alpha^n \det(A^2+B^2),$$

so

$$\alpha^n = \frac{\det(M\overline{M})}{\det(A^2+B^2)} = \frac{\det M \cdot \overline{\det M}}{\det(A^2+B^2)} = \frac{|\det M|^2}{\det(A^2+B^2)}$$

is a real number, since $\det(A^2+B^2) \neq 0$.

Now note that

$$\alpha^n = \left(\frac{2}{\sqrt{3}}\, e^{-\pi/6i}\right)^n = \left(\frac{2}{\sqrt{3}}\right)^n e^{-n\pi/6i},$$

and this number is real iff $-n\pi/6$ is an integral multiple of π. Thus, n must be a multiple of 6.

This problem has appeared in other forms; the most recent one was probably as problem #2998 in the undergraduate mathematics journal *Crux Mathematicorum*, whose solution was given in the December 2005 issue of that journal.