

10846. Proposed by Józef Przytycki, The George Washington University, Washington, DC. A lattice simple closed curve in the plane is a simple closed curve all of whose points have an integer coordinate. The area $A(C)$ enclosed by such a curve C is always a positive integer. A *shrinking* move on a lattice simple closed curve C is one that replaces C with another such curve D whose interior is inside C and with $A(D) = A(C) - 1$. Is it always possible to transform a lattice simple closed curve to a 1-by-1 square via shrinking moves?

Solution by Christopher Carl Heckman: The answer is yes. In fact, a stronger statement will be proven: Namely, that if $A(C) \geq 2$, then there are at least *two* lattice simple closed curves D_1, D_2 whose interiors are inside of C and $A(D_1) = A(D_2) = A(C) - 1$. (Note that two is best possible, because C could be the boundary of $(0, n) \times (0, 1)$, for any $n \geq 2$.)

Before this is to be proven, some terminology is introduced. A *block* is the set $B_{m,n} = (m, m+1) \times (n, n+1)$, where m and n are integers; note that any block is entirely inside of C or entirely outside of C . Two blocks (inside C) will be said to be *adjacent* if their closures intersect in a nontrivial line segment. The *degree* of a block is the number of blocks to which it is adjacent.

Given a block B (whose closure intersects C), $C \triangle B$ will denote the lattice closed curve whose segments are those of $C \cup \partial B$ which are in exactly one of C and ∂B . Note that $C \triangle B$ is simple iff $C \cap \partial B$ is connected.

We now prove the following statement. Applying it repeatedly provides a sequence of shrinking moves which end in a 1-by-1 square.

Theorem. *Let C be a lattice simple closed curve whose interior consists of at least two blocks. Then there exist two blocks B_1 and B_2 such that $C \triangle B_i$ is a lattice simple closed curve with area equal to $A(C) - 1$, for $i = 1, 2$.*

Proof: The proof is by induction on the number of blocks inside of C (equivalently, the area of C).

The curve C has no blocks with degree zero; otherwise, C would have to have area one. So now suppose that C has a block B of degree one. Then $C' = C \triangle B$ is a lattice simple closed curve whose area is one less than that of C . By induction, either C' has exactly one block (in which case, C satisfies the theorem), or C' has two blocks B_1 and B_2 such that $C' \triangle B_i$ is a lattice simple closed curve with area equal to $A(C') - 1 = A(C) - 2$. At least one of these blocks (wolog B_1) is not adjacent to B . Then $C \triangle B_1$ is a lattice closed curve whose area is $A(C) - 2 + 1 = A(C) - 1$, and $C \triangle B_1$ is simple because B_1 is inside C and intersects B in at most one vertex. Thus C satisfies the theorem.

Now suppose that C has no blocks of degree one. Choose the smallest integer m such that

$$((m, m+1) \times Z) \cap C \neq \emptyset.$$

Such an integer m exists, because C has finite area. Choose an integer n such that $B_{m,n}$ is inside C . Now let n^+ be the largest integer such that the blocks $B_{m,n}, B_{m,n+1}, \dots, B_{m,n^+}$ are all inside C , and let n^- be the smallest integer such that $B_{m,n^-}, B_{m,n^-+1}, \dots, B_{m,n}$ are inside C . Then $n^+ \geq n \geq n^-$, and $n^+ \neq n^-$; the latter follows because otherwise $B_{m,n}$ would have degree one.

If both of $C \triangle B_{m,n^+}$ and $C \triangle B_{m,n^-}$ are lattice simple closed curves, then C satisfies the theorem. Hence, we may assume that $C \triangle B_{m,n^+}$ is not (by symmetry).

Note that $B = B_{m,n^+}$ has degree at most two, because B_{m-1,n^+} and B_{m,n^++1} are not inside C , by construction of m and n^+ . Since there are no blocks with degree one, B has degree two. The reason that $D = C \triangle B$ fails to be a lattice simple closed curve is because the curve D intersects itself, at the point $(m+1, n^+)$. This is because $C \cap \partial B$ must contain two adjacent segments. For $C \triangle B$ to not be simple would require that ∂B intersects C elsewhere along C . This intersection cannot be along the other two segments incident with B , because B would then have degree one. Hence the place where ∂B intersects C is the single point $(m+1, n^+)$.

The curve D thus consists of the union of two lattice simple closed curves D_1 and D_2 ; hence induction can be applied to them. Neither of D_1 or D_2 is a single block, because then C would have a block of degree one. Thus D_1 has two blocks B'_1 and B'_2 such that $D_1 \triangle B'_i$ is a lattice simple closed curve, for $i = 1, 2$; D_2 contains similar blocks B''_1 and B''_2 . At least one of B'_1 and B'_2 (and one of B''_1 and B''_2) is not adjacent to B in C . For these two blocks B_1 and B_2 , $C \triangle B_i$ is a lattice simple closed curve, as desired. \square