11187. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL.* Find a closed formula for the number of ways to tile a 4 by n rectangle with 1 by 2 dominoes.

Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ: The answer can be expressed in two forms; one is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{n-1} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \tag{1}$$

and the other is

$$\begin{aligned} &\left(\tfrac{1}{4} - \tfrac{\sqrt{29}}{116} - \tfrac{\sqrt{58}\sqrt{7-\sqrt{29}}}{116}\right)\left(\tfrac{1-\sqrt{29}+\sqrt{14-2\sqrt{29}}}{4}\right)^n + \left(\tfrac{1}{4} - \tfrac{\sqrt{29}}{116} + \tfrac{\sqrt{58}\sqrt{7-\sqrt{29}}}{116}\right)\left(\tfrac{1-\sqrt{29}-\sqrt{14-2\sqrt{29}}}{4}\right)^n \\ &+ \left(\tfrac{1}{4} + \tfrac{\sqrt{29}}{116} + \tfrac{\sqrt{58}\sqrt{7+\sqrt{29}}}{116}\right)\left(\tfrac{1+\sqrt{29}+\sqrt{14-2\sqrt{29}}}{4}\right)^n + \left(\tfrac{1}{4} + \tfrac{\sqrt{29}}{116} - \tfrac{\sqrt{58}\sqrt{7+\sqrt{29}}}{116}\right)\left(\tfrac{1+\sqrt{29}-\sqrt{14-2\sqrt{29}}}{4}\right)^n. \end{aligned} \tag{2}$$

Both are obtained after finding several recurrences that solve this, and some related, problems.

In this discussion, the $4 \times n$ rectangle will have a width of 4 squares, and its squares will be indexed by $\{1, 2, 3, 4\} \times \{1, 2, \ldots, n\}$. An *arch of height* n will be the $4 \times n$ rectangle with the squares $(1, n)$ and $(4, n)$ removed. A *tower of height* n will be the $4 \times n$ rectangle with the squares $(2, n)$ and $(3, n)$ removed. A *leaner of height* n will be a $4 \times n$ rectangle with the squares $(1, n)$ and $(2, n)$ removed. Note that arches, towers, and leaners represent all possible ways to remove two squares of "opposite colors" from the top row of a rectangle of height n, once the squares have been colored like a chessboard.

The sequences R_n, A_n, T_n, and L_n will represent, respectively, the number of ways to tile (using 1×2 dominoes) a rectangle, an arch, a tower, and a leaner of height n. Brute force yields the equations $R_1 = 1$, $A_1 = 1$, $T_1 = 0$, $L_1 = 1$ and $R_2 = 5$.

Now we consider A_n, the number of ways to tile an arch of height n. For such a tiling, the two squares $(2, n)$ and $(3, n)$ are either covered by the same domino, or by different dominoes. If they are covered by the same domino, the rest of the tiling tiles a rectangle of height $n-1$. Since the number of ways to tile a rectangle of height $n-1$ is R_{n-1}, this is also the number of tilings of an arch of height n where the two squares at the top are covered by the same domino. If the two squares $(2, n)$ and $(3, n)$ are covered by two dominoes, the rest of the tiling tiles a tower of height $n-1$, and there are T_{n-1} of these tilings. Consequently,

$$A_n = R_{n-1} + T_{n-1}, \qquad \text{for } n \geq 2.$$

Similar reasoning leads to the rest of the equations given below:

$$\begin{aligned} R_n &= R_{n-2} + 2L_{n-1} + A_{n-1} + R_{n-1} \\ A_n &= R_{n-1} + T_{n-1} \\ T_n &= A_{n-1} \\ L_n &= L_{n-1} + R_{n-1} \end{aligned} \tag{3}$$

as long as the subscripts are all positive integers.

To get form (1) for R_n, we let $X_n = \begin{pmatrix} R_n & A_n & T_n & L_n & R_{n-1} \end{pmatrix}^{\mathrm{T}}$; then the system (3) can be written as the matrix equation

$$X_n = \begin{pmatrix} R_n \\ A_n \\ T_n \\ L_n \\ R_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} R_{n-1} \\ A_{n-1} \\ T_{n-1} \\ L_{n-1} \\ R_{n-2} \end{pmatrix} = MX_{n-1}, \tag{4}$$

where $n \geq 2$, and M is the 5×5 matrix in (4).

Iterating this matrix equation yields

$$X_n = M^{n-1} X_1,$$

and to extract R_n, we can multiply X_n on the left by $(1 \quad 0 \quad 0 \quad 0 \quad 0)$. Since we found values for R_1, A_1, T_1, and L_1, and $R_0 = 1$ (since the "null tiling" is a tiling, and this value also continues the recursions in (3)), $X_1 = (1 \quad 1 \quad 0 \quad 1 \quad 1)^{\mathrm{T}}$, and we obtain (1) above.

Form (2) above can be obtained by diagonalizing the matrix M, which turns out to have distinct eigenvalues, and by using the fact that $(PDP^{-1})^k = PD^kP^{-1}$. It can also be obtained by manipulating the recurrences themselves, which is illustrated below.

We start with the equation $T_n = A_{n-1}$, which allows us to remove the T's from the equations in (3). Since this equation also implies that $T_{n-1} = A_{n-2}$, we now have the system of recurrences

$$\begin{aligned} R_n &= R_{n-2} + 2L_{n-1} + A_{n-1} + R_{n-1}, \\ A_n &= R_{n-1} + A_{n-2}, \\ L_n &= L_{n-1} + R_{n-1}. \end{aligned}$$

Then we can eliminate the R's with the second recurrence, which implies that $R_{n-1} = A_n - A_{n-2}$. After this replacement, we get

$$\begin{aligned} 2L_{n-1} &= A_{n+1} - A_n - 3A_{n-1} + A_{n-2} + A_{n-3}, \\ L_n &= L_{n-1} + A_n - A_{n-2}. \end{aligned}$$

Then we eliminate the L's, using the first equation, and arrive at

$$A_{n+2} - 2A_{n+1} - 4A_n + 4A_{n-1} + 2A_{n-2} - A_{n-3} = 0, \tag{5}$$

with suitable initial conditions.

Now we are on familiar ground, and we can use the following method to find a formula for A_n; it is discussed in many introductory combinatorics books, such as Rosen's *Discrete Mathematics and Its Applications*. The equation (5) is a homogeneous linear recurrence with constant coefficients, so we look for a solution of the form $A_n = r^n$, and this substitution leads to the characteristic equation

$$r^5 - 2r^4 - 4r^3 + 4r^2 + 2r - 1 = 0,$$

which has five real roots:

$$1, \frac{1}{2}\left(\frac{1-\sqrt{29}}{2} \pm \sqrt{\frac{7-\sqrt{29}}{2}}\right), \frac{1}{2}\left(\frac{1+\sqrt{29}}{2} \pm \sqrt{\frac{7+\sqrt{29}}{2}}\right),$$

and which will be denoted $r_1, \ldots, r_5$.

Not only is $A_n = r_i^n$ a solution to (5) for any i, but so is any sequence of the form

$$C_1 r_1^n + C_2 r_2^n + \cdots + C_5 r_5^n,$$

for arbitrary constants C_i. Because L_n and R_n can be written as linear combinations of A_k for a finite number of values of k, L_n and R_n will also be of this form. Once we calculate R_1 through R_5 (using the recurrence) and setting

$$R_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_5 r_5^n,$$

for $n = 1, \ldots, 5$, we obtain a system of linear equations in the c_i's which has a unique solution. Substituting these values in for the c_i's yields (2) above. The details are left as an exercise for the overzealous reader.

This procedure (generating and solving a recurrence) has been used to show that the number of tilings of a 2 by n rectangle with 1 by 2 dominoes is a shifted Fibonacci sequence.