**11426.** *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Find

$$\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)}$$

where $\Gamma$ denotes the usual gamma function, given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$.

*Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ*: Using three identities that $\Gamma$ satisfies, as well as a trigonometric identity, it can be shown that this expression equals 2. Three well-known identities involving the Gamma function are the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \equiv c_z,$$

the duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \equiv a_z\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

and the reduction formula

$$\Gamma(1+z) = z\cdot\Gamma(z),$$

where the notations $c_z$ and $a_z$ have been introduced for legibility's sake.

Then

$$\begin{aligned}
\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)} &= \frac{[\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)]^2}{c_{3/14}\cdot c_{5/14}\cdot c_{13/14}} \\
&= \frac{1}{c_{3/14}\cdot c_{5/14}\cdot c_{13/14}}\cdot\left[\frac{\Gamma(1/7)}{a_{1/14}\cdot\Gamma(4/7)}\cdot\frac{\Gamma(9/7)}{a_{9/14}\cdot\Gamma(8/7)}\cdot\frac{\Gamma(11/7)}{a_{11/14}\cdot\Gamma(9/7)}\right]^2 \\
&= \frac{(a_{1/14}\cdot a_{9/14}\cdot a_{11/14})^{-2}}{c_{3/14}\cdot c_{5/14}\cdot c_{13/14}}\cdot\left[\frac{\Gamma(1/7)}{\Gamma(4/7)}\cdot\frac{2/7\cdot\Gamma(2/7)}{1/7\cdot\Gamma(1/7)}\cdot\frac{4/7\cdot\Gamma(4/7)}{2/7\cdot\Gamma(2/7)}\right]^2 \\
&= \frac{16\cdot(a_{1/14}\cdot a_{9/14}\cdot a_{11/14})^{-2}}{2c_{3/14}\cdot c_{5/14}\cdot c_{13/14}} = 16\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right).
\end{aligned}$$

The exact value of this expression can be found using a product-of-sines formula:

$$\prod_{k=1}^{\lfloor n/2\rfloor}\sin\left(\frac{k\pi}{n}\right) = \sqrt{\frac{n}{2^{n-1}}},$$

substituting the values of $n=2$, $n=7$, and $n=14$:

$$\begin{aligned}
\sin\left(\frac{\pi}{2}\right) &= \sqrt{\frac{2}{2^1}} = 1 \\
\sin\left(\frac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{3\pi}{7}\right) &= \sqrt{\frac{7}{2^6}} = \frac{\sqrt{7}}{8} \\
\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{2\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\cdots\sin\left(\frac{7\pi}{14}\right) &= \sqrt{\frac{14}{2^{14}}} = \frac{\sqrt{14}}{2^7}
\end{aligned}$$

which imply that $\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right) = \frac{1}{8}$, and the claim holds. $\Longrightarrow$

Now for a more interesting question: generalizing the problem. The original expression given can be written in the form

$$P(N,g) = \prod_{i=0}^{\operatorname{ord}(g)-1} \Gamma\left(\frac{g^i}{N} \bmod 1\right)^{(-1)^i}$$

(with $g = 3$ and $N = 14$, and $\operatorname{ord}(g)$ the smallest positive integer $r$ such that $g^r \equiv 1 \pmod{N}$), and it also can be shown to be equal to a formula which does not involve the Gamma function at all.

The fact that $g$ is a generator of the group $U_N$ (the set of all positive integers less than $N$ relatively prime to $N$) is necessary for full cancellation; hence $\operatorname{ord}(g) = \phi(N)$, the Euler totient function.

Note also that the question of whether the expression $P(N,g)$ can be made into a Gamma-free expression does not change if we drop the "mod 1" part of the formula, although its value will. Also, different generators can result in different values of $P(N,g)$.

Now we turn to new results (as these types of expressions do not seem to be in the literature). We start with a well-known fact:

**Theorem.** *The group $U_N$ is cyclic (it has a generator) if $N = 2$, $4$, or of the form $p^k$ or $2p^k$, where $p$ is an odd prime and $k$ is a positive integer.*

Also, $\phi(N)$ is even if $N \geq 3$. By symmetry arguments it follows that $g^{\phi(N)/2} = -1$. Finally, if $g$ is a generator of $U_N$, we define $\log_g m$ to be the unique integer $\lambda$ between 0 and $\phi(N)-1$ such that $g^\lambda = m$.

**Theorem 1.** *If $N = p^k$ or $2p^k$, where $p \equiv 1 \pmod{4}$, then*

$$P(N,g) = \prod_{\substack{1\le k<N/2\\ \gcd(k,N)=1}} \left[\sin\left(\frac{k\pi}{N}\right)\right]^{(-1)^{1+\log_g k}}.$$

*P*roof: $\log_g(-1) = \phi(N)/2$, which is an even integer by assumption. Thus the factors $\Gamma\left(\frac{1}{N}\right)$ and $\Gamma\left(\frac{N-1}{N}\right)$ appear in the numerator of $P(N,g)$. Also, if $k$ is an integer (in the appropriate range), $\Gamma\left(\frac{k}{N}\right)$ and $\Gamma\left(\frac{N-k}{N}\right)$ will appear either both in the numerator or both in the denominator, since

$$\log_g(N-k) \equiv \log_g(-1) + \log_g k \pmod{\phi(N)} \equiv \log_g k \pmod{2}.$$

Thus, the inversion formula can be used to simplify $P(N,g)$, resulting in the expression above. □

The cancellation that shows up in the original problem shows up in the following lemma, which will allow us to evaluate other expressions $P(N,g)$. Also, we define

$$h(x) = \begin{cases} 1, & \text{if } x < 1 \\ x-1, & \text{if } x > 1 \end{cases}$$

and $m_i = \frac{g^i}{N} \bmod 1$, the fractional part of $\frac{g^i}{N}$.

**Lemma 2.** *Suppose that $p$ is an odd prime, $k$ is a positive integer, and $(g^2)^j \equiv 2 + p^k \pmod{2p^k}$ for some positive integer $j$. Then*

$$\prod_{i=0}^{\phi(2p^k)/2-1} \Gamma\left(\frac{g^{2i}}{2p^k} \bmod 1\right) = \prod_{i=0}^{\phi(2p^k)/2-1} \frac{1}{a_{g^{2i}/2p^k \bmod 1}} \cdot \frac{h(2\cdot(g^{2i}/2p^k \bmod 1))}{h((g^{2i}/2p^k \bmod 1) + \frac{1}{2})},$$

*where $a_x$ is defined as in the duplication formula.*

*P*roof: The duplication formula implies that

$$\Gamma\left(m_{2i}\right) = \frac{1}{a_{m_{2i}}} \cdot \frac{\Gamma\left(2m_{2i}\right)}{\Gamma\left(m_{2i}+\frac{1}{2}\right)}.$$

Then the reduction identity is applied if $m_{2i} > \dfrac{1}{2}$; this implies that

$$\Gamma\left(m_{2i}\right) = \frac{h(2m_{2i})}{a_{m_{2i}} \cdot h(m_{2i}+\frac{1}{2})} \cdot \frac{\Gamma\left(\frac{g^{2i}}{p^k} \bmod 1\right)}{\Gamma\left(\left(\frac{g^{2i}}{2p^k}+\frac{1}{2}\right) \bmod 1\right)}.$$

Finally, the existence of a $j$ such that $(g^2)^j \equiv 2+p^k \pmod{2p^k}$ implies that the numerator of the rewritten form of $\Gamma\left(m_{2i}\right)$ will cancel with the denominator of $\Gamma\left(m_{2(i+j)}\right)$, where $i+j$ is calculated modulo $\phi(2p^k)/2$: If $(g^2)^j \equiv 2+p^k \pmod{2p^k}$, then $2 \equiv g^{2j}+p^k \pmod{2p^k}$, and multiplying both sides of this equation by $g^{2i}$ yields

$$2g^{2i} \equiv g^{2i}\cdot g^{2j} + g^{2i}\cdot p^k \pmod{2p^k} \equiv g^{2(i+j)} + p^k \pmod{2p^k},$$

since $g$ (and hence $g^{2i}$) must be odd. Then, dividing both sides of this equation by $2p^k$,

$$2m_{2i} \equiv \frac{g^{2i}}{p^k} \equiv \frac{g^{2(i+j)}}{2p^k} + \frac{1}{2} \equiv m_{2i} + \frac{1}{2} \pmod{1},$$

as claimed. □

**Theorem 3.** *Suppose that $p$ is an odd prime such that $p \equiv 3 \pmod{4}$, $k$ is a positive integer, and $(g^2)^j \equiv 2+p^k \pmod{2p^k}$ for some positive integer $j$. Then*

$$P(2p^k, g) = \prod_{i=0}^{\phi(2p^k)/2-1} \left[c_{g^{2i}/2p^k \bmod 1}\right]^{-1} \cdot \left[\prod_{i=0}^{\phi(2p^k)/2-1} \frac{h(2m_{2i})}{a_{m_{2i}} \cdot h(m_{2i}+\frac{1}{2})}\right]^2,$$

*where $a_x$ is defined as in the duplication formula, and $c_x$ is defined as in the inversion formula.*

*P*roof: Use the inversion formula and Lemma 2, as was done for the original problem. □

**Corollary 4.** *Under the same assumptions as Theorem 3,*

$$P(2p^k, g) = \prod_{i=0}^{\phi(2p^k)/2-1} \sin\left(m_{2i}\pi\right) \cdot 16^{-m_{2i}} \cdot h^*\left(m_{2i}\right),$$

*where* $h^*(x) = \begin{cases} 4, & \text{if } x < \dfrac{1}{2} \\ 16, & \text{if } x > \dfrac{1}{2} \end{cases}$ *and* $m_i = \dfrac{g^i}{2p^k} \bmod 1.$

*P*roof: Substitution and cleanup. (More simplification can probably be done, but I ran out of time before I could simplify $\sum m_{2i}$ or count how many $m_{2i}$'s are less than $\frac{1}{2}$.) □

Note that this is a true generalization of the original result, since $7 \equiv 3 \pmod{4}$ and $(g^2)^1 \equiv (3^2)^1 \equiv 2+7^1 \equiv 2+p^k \pmod{2p^k}$.

"Where have all the Gammas gone?"