

11424. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Find the number of bit strings of length n in which the number of 00 substrings is equal to the number of 11 substrings. For example, when $n = 4$ we have 4 such bit strings: 0011, 0101, 1010, and 1100.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: When $n \geq 1$, the number a_n of such strings is given below.

$$a_n = \begin{cases} 2 \cdot \binom{n-2}{n/2-1}, & \text{if } n \text{ is even} \\ 2 \cdot \binom{n-2}{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases} = \binom{n-2}{\lceil n/2 \rceil - 1}$$

[Here, I am assuming that the occurrences of the 00's and 11's are not necessarily disjoint; in particular, 00011011 is to be counted.]

Suppose we are given a string among those to be counted. Define z_i to be the number of zeros in the i th run of zeros, o_i to be the number of ones in the i th run of ones, $z = \sum_i z_i$, and $o = \sum_i o_i$. (For instance, for the string 00011011, $z_1 = 3$, $z_2 = 1$, and $o_1 = o_2 = 2$.) The number of 00 substrings in the given string then becomes $\sum_i (z_i - 1)$ and the number of 11 substrings is $\sum_i (o_i - 1)$.

If the given string of length n begins with a 0 and ends in a 1, then the number of runs of 0s and 1s is the same; call it k . Then the condition that the number of 00 substrings equals the number of 11 substrings becomes

$$\begin{aligned} \sum_{i=1}^k (z_i - 1) &= \sum_{i=1}^k (o_i - 1) \\ z - k &= o - k, \end{aligned}$$

or $z = o$. This, combined with the condition that $z + o = n$, implies that $z = o = n/2$ and that n is even. Similarly, if a binary string of length n has $z = o$ and begins with 0 and ends with 1, then the number of 00 substrings equals the number of 11 substrings. Thus the number of desired strings of length n of the form $0*1$ is $\binom{n-2}{n/2-1}$. This is also the number of desired strings of the form $1*0$, by symmetry.

Now, if the string of length n begins and ends with a 0, then it has one more 0-run than 1-runs. The balancing condition then becomes

$$\begin{aligned} \sum_{i=1}^{k+1} (z_i - 1) &= \sum_{i=1}^k (o_i - 1) \\ z - (k + 1) &= o - k, \end{aligned}$$

or $z = o + 1$. Solving this equation, along with $z + o = n$, yields $o = \frac{n-1}{2} = z - 1$. So the string has $(n-1)/2$ ones, the other entries are zeros, and there are thus $\binom{n-2}{(n-1)/2}$ such strings. Furthermore, n is odd. The same results follow for strings which begin and end with 1. Combining these results produces the formula above (with the convention that $\binom{-1}{0} = 1$).

The number of binary strings of length n with the same number of 01 and 10 substrings is 2^{n-1} ; these are just the strings that begin and end with the same bit. The number of binary strings of length n with the same number of 00 and 01 substrings is an interesting problem. The first few terms in this sequence are 1, 2, 2, 3, 6, 9, 15, 30, 54, 97, 189, 360, 675, 1304, 2522, 4835, 9358, 18193, 35269, 68568, 133737, 260802, 509132; this sequence does not (yet) appear in the Online Encyclopedia of Integer Sequences. Once this problem is solved, all variations have been solved, by symmetry.