11406. *Proposed by A. A. Dzhumadil'daeva, Almaty, Republics Physics and Mathematics School, Almaty, Kazakhstan.* Let $n!!$ denote the product of all positive integers not greter than n and congruent to n mod 2, and let $0!! = (-1)!! = 1$. Thus, $7!! = 105$ and $8!! = 384$. For positive integer n, find

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!(2(n-i)-1)!!$$

in closed form.

Two solutions by Christopher Carl Heckman, Arizona State University, Tempe, AZ: The sum equals $2^n \cdot n!$ for all $n \geq 1$.

Solution #1. To prove this, we use generating functions: We let $f_i = (2i-1)!!$ and $f(x)$ the exponential generating function of f_i; then the sum above is the coefficient of $\frac{x^n}{n!}$ in $\bigl(f(x)\bigr)^2$, by the definition of the product of two power series.

To get a closed-form for $f(x)$, note that

$$\frac{f_i}{i!} = \frac{1}{i!} \cdot \frac{(2i)!}{2^i \cdot i!} = \frac{1}{2^i} \binom{2i}{i}.$$

Thus,

$$f(x) = \sum_{i=0}^{\infty} \frac{f_i}{i!} x^i = \sum_{i=0}^{\infty} \binom{2i}{i} \left(\frac{x}{2}\right)^i = \frac{1}{\sqrt{1 - 4 \cdot \left(\frac{x}{2}\right)}} = \frac{1}{\sqrt{1-2x}}.$$

The closed form for $\sum_{i=0}^{\infty} \binom{2i}{i} y^i$ is well-known; for instance, it is formula (2.5.11) in Wilf's *generatingfunctionology*.

Then

$$\bigl(f(x)\bigr)^2 = \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} \frac{2^n \cdot n!}{n!} x^n,$$

and the coefficient of $\frac{x^n}{n!}$ is $2^n \cdot n!$, which solves the given problem.

Solution #2. To prove this, we use Sister Celine's method, as described in Marko Petkovšek, Herbert Wilf and Doron Zeilberger's *A=B*. Let $f(n,i) = \binom{n}{i}(2i-1)!!(2(n-i)-1)!!$; now the goal is to find a recurrence that $f(n,i)$ satisfies. Celine's method, coded in Maple, gives the following:

$$4(n+2)^2 f(n,i) + (-2n-3) f(n+1,i) + (-2n-3) f(n+1,i+1) + f(n+2,i+1) = 0.$$

(This equation can be verified by hand by showing that

$$4(n+2)^2 \cdot \frac{f(n,i)}{f(n,i)} + (-2n-3) \cdot \frac{f(n+1,i)}{f(n,i)} + (-2n-3) \cdot \frac{f(n+1,i+1)}{f(n,i)} + \frac{f(n+2,i+1)}{f(n,i)} = 0.)$$

Now, sum this recurrence over all values of i, to get

$$4(n+2)^2 F(n) + (-2n-3) F(n+1) + (-2n-3) F(n+1) + F(n+2) = 0,$$

where $F(n) = \sum_i f(n,i)$. This is a second-order recurrence, so if a formula $F(n)$ satisfies it, and $F(0) = 1$ and $F(1) = 2$, then $F(n) = \sum_{i=0}^{n} \binom{n}{i}(2i-1)!!(2(n-i)-1)!!$ for all n. The formula $F(n) = 2^n \cdot n!$ meets all the conditions, so the sum equals $2^n \cdot n!$.