11394. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* A fair coin is tossed n times, with $n \geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: Define the following random variables:

$$H_k = \begin{cases} 1, & \text{if the } n\text{th toss is heads and the } (n+1)\text{st toss is tails} \\ 0, & \text{otherwise} \end{cases}$$

$$T_k = \begin{cases} 1, & \text{if the } n\text{th toss is tails and the } (n+1)\text{st toss is heads} \\ 0, & \text{otherwise} \end{cases}$$

Then $X = H_1 + T_{n-1} + \sum_{j=1}^{n-2} H_{j+1}T_j$ and $R = 1 + \sum_{i=1}^{n-1} H_i + \sum_{i=1}^{n-1} T_i$.

The product of any H_i's and T_j's can only be 0 or 1, so the expected value of any such expression is the probability that it is one. Since expectation is linear, even if the variables are not independent,

$$E(X) = E(H_1) + E(T_{n-1}) + \sum_{j=1}^{n-2} E(H_{j+1}T_j) = \frac{1}{4} + \frac{1}{4} + (n-2)\cdot\frac{1}{8} = \frac{n+2}{8} \tag{1}$$

$$E(R) = E(1) + \sum_{i=1}^{n-1} E(H_i) + \sum_{i=1}^{n-1} E(T_i) = 1 + (n-1)\cdot\frac{1}{4} + (n-1)\cdot\frac{1}{4} = \frac{n+1}{2} \tag{2}$$

Now we need $E(XR) = E\left[\left(H_1 + T_{n-1} + \sum_{j=1}^{n-2} H_{j+1}T_j\right)\left(1 + \sum_{i=1}^{n-1} H_i + \sum_{i=1}^{n-1} T_i\right)\right]$, which is

$$\begin{aligned} E(X) + \sum_{i=1}^{n-1} E(H_1H_i) + \sum_{i=1}^{n-1} E(H_1T_i) + \sum_{i=1}^{n-1} E(T_{n-1}H_i) + \sum_{i=1}^{n-1} E(T_iT_{n-1}) \\ + \sum_{j=1}^{n-2}\sum_{i=1}^{n-1} E(H_iT_jH_{j+1}) + \sum_{j=1}^{n-2}\sum_{i=1}^{n-1} E(T_iT_jH_{j+1}). \end{aligned} \tag{3}$$

But

$$E(H_1H_i) = \begin{cases} 1/4, & \text{if } i = 1 \\ 0, & \text{if } i = 2 \\ 1/16, & \text{otherwise} \end{cases} \quad \text{so} \quad \sum_{i=1}^{n-1} E(H_1H_i) = \frac{1}{4} + (n-3)\cdot\frac{1}{16} = \frac{n+1}{16}. \tag{4}$$

Similarly,

$$E(H_1T_i) = \begin{cases} 0, & \text{if } i = 1 \\ 1/8, & \text{if } i = 2 \\ 1/16, & \text{otherwise} \end{cases} \quad \text{so} \quad \sum_{i=1}^{n-1} E(H_1T_i) = \frac{1}{8} + (n-3)\cdot\frac{1}{16} = \frac{n-1}{16}. \tag{5}$$

Now fix j between 1 and $n-2$. Then

$$E(H_iT_jH_{j+1}) = \begin{cases} 1/32, & \text{if } i < j-1 \text{ or } i > j+2 \\ 1/16, & \text{if } i = j-1 \\ 0, & \text{if } i = j \text{ or } i = j+2 \\ 1/8, & \text{if } i = j+1 \end{cases} \tag{6}$$

so

$$\sum_{i=1}^{n-1} E(H_iT_jH_{j+1}) = \frac{1}{16} + \frac{1}{8} + (n-1-4)\cdot\frac{1}{32} = \frac{n+1}{32},$$

unless $j = 1$, in which case the sum is actually

$$\frac{1}{8} + \left(((n-1)-3) \cdot \frac{1}{32}\right) = \frac{n}{32},$$

since there is a $\frac{1}{32}$ term instead of $\frac{1}{16}$, as $i = 0$ when $i = j - 1$.

So

$$\sum_{j=1}^{n-2} \sum_{i=1}^{n-1} E(H_i T_j H_{j+1}) = (n-3) \cdot \frac{n+1}{32} + \frac{n}{32}. \tag{7}$$

Lastly, similar arguments or an appeal to symmetry shows that

$$\sum_{i=1}^{n-1} E(H_i T_{n-1}) = \frac{n+1}{16}, \quad \sum_{i=1}^{n-1} E(T_i T_{n-1}) = \frac{n-1}{16}, \quad \text{and} \quad \sum_{j=1}^{n-2} \sum_{i=1}^{n-1} E(T_i T_j H_{j+1}) = (n-3) \cdot \frac{n+1}{32} + \frac{n}{32}. \tag{8}$$

Finally, combining equations (1)–(5), (7), and (8), with the fact that the covariance of X and R is $E(XR) - E(X)E(R)$, produces the desired result.