11377. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX, and Lionel Levine, Massachusetts Institute of Technology, Cambridge, MA.* Given a monic polynomial p of degree n with complex coefficients, let A_p be the $(n+1)\times(n+1)$ matrix with $p(-i+j)$ in position (i,j), and let D_p be the determinant of A_p. Show that D_p depends only on n, and find its value in terms of n.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: $D_p = (n!)^{n+1}$. To see this, perform the following matrix operations on A_p, in this order. (Note that $C(n,k)$ is the binomial coefficient.)

(1) For each j from 0 to n in increasing order, and for each k between $j+2$ and $n+1$, subtract $C(k-1,j)$ times row $j+1$ from row k.
(2) For each j from 0 to n in increasing order, and for each k between $j+2$ and $n+1$, subtract $C(k-1,j)$ times column $j+1$ from column k.

Since subtracting a multiple of one row (or column) from another row (resp. column) does not change the determinant, the matrix obtained (which will be called B_p) has the same determinant as A_p.

After a lengthy but straightforward calculation, it can be shown that the entry in position (i,j) of B_p is

$$b_{i,j} = \sum_{k=0}^{i+j-2} (-1)^{k+j+1} C(i+j-2,k)p(k+1-i).$$

This expression can be written more compactly. Given a function f, define Δf to be the function $(\Delta f)(x) = f(x+1) - f(x)$. Then $b_{i,j}$ is seen to be $(-1)^{i+1}\Delta^{i+j-2}p(j-i)$, where Δ^k indicates that application of the Δ operator k times; this can be proven by induction on $i+j$.

It is well-known that, if p is a polynomial of degree n, then $\Delta^n p(x) = n!\cdot a_n$, where a_n is the coefficient of x^n in p (which is equal to one here), and $\Delta^k p(x) = 0$ for all $k > n$.

This means $b_{i,j} = (-1)^{i+1}\cdot n!$ if $i+j = n+2$, i.e., if (i,j) lies on the diagonal of B_p connecting $(1,n+1)$ and $(n+1,1)$, and $b_{i,j} = 0$ if $i+j > n+2$. This is enough to calculate the determinant of B_p (and hence of A_p). Swap row k with row $n+2-k$, for all $k \le \left\lfloor \frac{n+1}{2} \right\rfloor$ to get C_p; then the determinant of C_p is $(-1)^{\lfloor (n+1)/2 \rfloor}$ times the determinant of B_p.

Finally, C_p is an upper-triangular matrix where $\left\lfloor \frac{n+1}{2} \right\rfloor$ entries are equal to $(-1)\cdot n!$, and the rest are equal to $n!$. Thus the determinant of C_p is $(-1)^{\lfloor (n+1)/2 \rfloor}(n!)^{n+1}$. The determinants of A_p and B_p are then

$$\frac{(-1)^{\lfloor (n+1)/2 \rfloor}(n!)^{n+1}}{(-1)^{\lfloor (n+1)/2 \rfloor}} = (n!)^{n+1},$$

as claimed.