

**11362.** *Proposed by David Callan, University of Wisconsin, Madison, WI.* A *bit string arc diagram* is an undirected graph in which the vertices are the positions in a single string of bits and the edges are called *arcs* due to the visual representation in which they are drawn joining positions in the string. To be a *good* diagram, arcs must occur only between unequal bits, and each bit may be the left endpoint of at most one arc. Thus, the first diagram is good but, for two reasons, the second is not.



There are six good diagrams on two bits, four with no arc and two with a single arc. How many good diagrams are there on  $n$  bits?

*Solution and generalization by Christopher Carl Heckman, Arizona State University, Tempe, AZ:* The answer to the given problem is  $(n + 1)!$ . More generally, if “bit string” is replaced by “a string containing characters from the set  $\{0, 1, \dots, k\}$ ”, with  $k \geq 2$ , the answer is

$$k \cdot \prod_{i=2}^n [(i-1)(k-1) + k] = \frac{k \cdot (k-1)^{n-1} \cdot \Gamma\left(\frac{nk-n+k}{k-1}\right)}{\Gamma\left(\frac{2k-1}{k-1}\right)},$$

where  $\Gamma$  is the Euler Gamma function.

This follows easily from the recursion

$$\begin{aligned} g_1^{(k)} &= k \\ g_n^{(k)} &= [(n-1)(k-1) + k] \cdot g_{n-1}^{(k)}, \quad n \geq 2 \end{aligned} \quad (\star)$$

where  $g_n^{(k)}$  is the number of good strings of length  $n$ .

To prove this, fix  $k$  and let  $G_n$  be the set of all good diagrams of length  $n$ . Then there is a natural map  $\phi: G_n \rightarrow G_{n-1}$  (if  $n \geq 2$ ): remove the vertex in the  $n$ th position. (According to the diagram above, the  $n$ th position is the left-most vertex.) Let  $\pi_k$  be the permutation  $(0\ 1\ 2\ \dots\ k-1)$ , for all  $k \geq 2$ , which will permute the characters in the string, and let  $G'_n(s) = \{s, \pi_k(s), \pi_k^2(s), \dots\}$ . (In particular,  $G'_3(011) = \{011, 122, 200\}$  if  $k = 3$ .) Then  $|G'_n(s)| = k$  for any string  $s$ ; the key to proving  $(\star)$  is to find out how many good diagrams in  $G_n$  map to an element  $D$  of  $G'_{n-1}(s)$ , where  $s$  is the string that labels the vertices of  $k$ . In what follows,  $s$  is fixed as well.

If there is no edge from  $n$  to the rest of a diagram, then the  $n$ th vertex can be labelled with any “digit”, so exactly  $k \cdot g_{n-1} = k^2 \cdot |G'_{n-1}(s)|$  good diagrams of this type map to  $G'_{n-1}(s)$ .

Now, suppose that the  $n$ th digit is  $i$ ; there are  $n-1$  possibilities for the vertex on the other end ( $j$ ). Now take a good diagram  $D$  in  $G'_{n-1}(s)$  and connect the  $n$ th vertex to the  $j$ th vertex to get a pre-image of  $D$ . Since the “digits” on the ends of the arcs must be unequal, there are exactly  $k-1$  diagrams  $D$  which can be chosen. Thus, if the  $n$ th digit is  $i$ , there are  $(n-1)(k-1)$  possible pre-images of elements in  $G'_{n-1}(s)$ , and since  $i$  can be any of  $k$  digits, there are exactly  $(n-1)(k-1)k \cdot |G'_{n-1}(s)|$  good diagrams which map into  $G'_{n-1}(s)$  where the  $n$ th digit is connected to another.

Summing over all strings  $s$ , we obtain

$$g_n = \sum [k^2 + (n-1)(k-1)k] |G'_{n-1}(s)| = [k^2 + (n-1)(k-1)k] \cdot \left(\frac{g_{n-1}}{k}\right) = [(n-1)(k-1) + k] \cdot g_{n-1},$$

which proves  $(\star)$ .