**11350.** *Proposed by Bhavana Deshpande, Poona College of Arts, Science & Commerce Camp, Pune, India, and M. N. Deshpande, Institute of Science, Nagpur, India.* Given a positive integer $n$ and an integer $k$ with $0 \le k \le n$, form a permutation $a = (a_1, \ldots, a_n)$ of $(1, \ldots, n)$ by choosing the first $k$ positions at random and filling the remaining $n-k$ positions in ascending order. Let $E_{n,k}$ be the expected number of left-to-right maxima. (For example, $E_{3,1} = 2$, $E_{3,2} = 11/6$, and $E_{4,2} = 13/6$.) Show that $E_{n+1,k} - E_{n,k} = 1/(k+1)$. (A left-to-right maximum occurs at $k$ when $a_j < a_k$ for all $j < k$.)

*Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ*: The result is clearly true if $k = 0$, since in each case the identity permutation is created and thus $E_{n,k} = n$, so we may assume $k > 0$.

First, of all, we show that

$$E_{n,k} = \frac{k!}{P(n,k)} \sum_{i=0}^{n-k} i \cdot \binom{n-i-1}{k-1} + \sum_{i=1}^{k} \frac{1}{i} = \frac{k!}{P(n,k)} \binom{n}{k+1} + \sum_{i=1}^{k} \frac{1}{i} = \frac{n-k}{k+1} + \sum_{i=1}^{k} \frac{1}{i}, \qquad (1)$$

where $P(n,k) = \dfrac{n!}{(n-k)!}$.

For $1 \le i \le k$, let $X_i$ be the random variable which is 1 if there is a left-to-right maximum at position $i$ and 0 otherwise, and $X_0$ the total number of left-to-right maxima in positions $k+1$ through $n$; clearly $E_{n,k} = \sum_{i=0}^{k} E(X_i)$. Since $X_i$ is a 0-1 variable, for $i > 0$, $E(X_i)$ is the probability that $X_i = 1$, which is $\dfrac{1}{i}$; this produces the second summation in the first part of (1).

To calculate $E(X_0)$, we consider the possible values of $X_0$. $X_0 = i$ occurs if $n - i$ is one of the first $k$ numbers, and $n, n-1, \ldots, n-(i-1)$ aren't, and the probability that this happens is $\dfrac{k!}{P(n,k)} \dbinom{n-i-1}{k-1}$. Summing up these probabilities from $i = 0$ to $i = n-k$ produces the first summation in the first part of (1). This proves the first part of (1).

To prove that $\sum_{i=0}^{n-k} i \cdot \binom{n-i-1}{k-1} = \binom{n}{k+1}$, a simple counting argument suffices; both sides are the number of subsets of $\{1, \ldots, n\}$ of size $k+1$, since the number of such subsets with the second-smallest number being $i+1$ is $i \cdot \dbinom{n-i-1}{k-1}$. The rest of (1) follows by conversion into factorials.

Using (1), the desired result follows immediately. Similarly, $E_{n,k+1} - E_{n,k} = -\dfrac{n-k-1}{(k+2)(k+1)}$.