11345. *Proposed by Roger Cuculière, France.* Find all nondecreasing functions f from R to R such that $f(x+f(y)) = f(f(x)) + f(y)$.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: The solutions are $f(x) = x$; $f(x) = 0$; $f(x) = a \cdot \left\lceil \frac{x-b}{a} \right\rceil$, for some constants $a > b \geq 0$; and $f(x) = a + a \cdot \left\lfloor \frac{x-b}{a} \right\rfloor$, for some constants $a \geq b > 0$. We start the proof with some substitutions into the given equation.

Substituting $x = 0$ and $y = 0$ yields $f(0) = 0$. Substituting $x = 0$ and leaving y yields $f(f(y)) = f(y)$, for all y. Lastly, if $a = f(y)$, then

$$f(x+a) = f(x+f(y)) = f(f(x)) + f(y) = f(x) + a. \qquad (\star)$$

Iterating $(\star)$ yields $f(na) = na$ for all integers n.

Now we show the functions above are the only possible ones. If f is identically zero, we're done; otherwise, there is an a which is in the range of f. We may assume that $a > 0$, since $f(-a) = -f(a) = -a$ by $(\star)$. If there are no elements in the range of f between 0 and a, then there is a b between 0 and a such that $f(x) = 0$ when $0 \leq x < b$, and $f(x) = a$ when $b < x \leq a$; this follows because f is nondecreasing on $[0, a]$. If $f(b) = 0$, then $(\star)$ implies that $f(x) = a \cdot \left\lceil \frac{x-b}{a} \right\rceil$, and since we cannot have $b = a$, f is of the form described above. If $f(b) = a$, then $f(x) = a + a \cdot \left\lfloor \frac{x-b}{a} \right\rfloor$ and clearly we must have $b > 0$.

Consequently, there are arbitrarily small positive real numbers ε in the range of f, and for which $f(\varepsilon) = \varepsilon$ (and hence $f(n\varepsilon) = n\varepsilon$). If $f(x) < x$ for some real number x, then choose an $\varepsilon < x - f(x)$ which is in the range of f. Now there is an integer n such that $x > n\varepsilon > f(x)$; then, since f is nondecreasing,

$$f(x) \geq f(n\varepsilon) = n\varepsilon,$$

which contradicts the choice of n. A similar proof shows that there are no values of x for which $f(x) > x$. Hence, $f(x) = x$, for all x, as desired. □

Remark: The only non*increasing* function that satisfies the equation above is $f(x) = 0$. This is because, for $(\star)$ to hold, a must be zero, and there can be no other elements in the range of f. The only strictly increasing function is $f(x) = x$.