

11343. Proposed by David Beckwith, Sag Harbor, NY. Show that when n is a positive integer,

$$\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}.$$

Solutions by Christopher Carl Heckman, Arizona State University, Tempe, AZ:

(1) **Solution using Counting.** Fix n , and define

$$A_{i,j} = \binom{n}{2i} \binom{n-2i}{j} \binom{2i}{i} 2^{n-2i-j}, \quad 0 \leq i \leq \frac{n}{2}, \quad 0 \leq j \leq 2i,$$

and zero for all other values of i and j . Then, using the Binomial Theorem,

$$\sum_{j \geq 0} A_{i,j} = \binom{n}{2i} \binom{2i}{i} \sum_{j=0}^{n-2i} \binom{n-2i}{j} \cdot 2^{(n-2i)-j} \cdot 1^j = \binom{n}{2i} \binom{2i}{i} 3^{n-2i}. \quad (1.1)$$

Converting the binomial coefficients to factorials, cancelling, and rearranging factors produces

$$\sum_{i \geq 0} A_{i,j} = \sum_{i \geq 0} \binom{n}{n-j} \binom{n-j}{2i} \binom{2i}{i} 2^{n-2i-j},$$

so it is sufficient to show that

$$\sum_{i \geq 0} \binom{k}{2i} \binom{2i}{i} 2^{k-2i} = \binom{2k}{k} \quad (1.2)$$

to prove the stated result, since then

$$\sum_{i \geq 0} \binom{n}{2i} \binom{2i}{i} 3^{2n-i} = \sum_{i \geq 0} \sum_{j \geq 0} A_{i,j} = \sum_{j \geq 0} \sum_{i \geq 0} A_{i,j} = \sum_{j \geq 0} \binom{n}{n-j} \binom{2(n-j)}{n-j} = \sum_{k \geq 0} \binom{n}{k} \binom{2k}{k}$$

follows by (1.1), (1.2), and the change of variable $k = n - j$.

To show (1.2), let S denote the set of all strings of length k with characters 0, 1, + and $-$, such that the number of +’s and $-$ ’s are the same, and T the subsets of $\{1, 2, \dots, 2k\}$ with size k . $|T|$ is clearly $\binom{2k}{k}$, and $|S|$ can be calculated by considering the number of $+/-$ pairs and summing over all possible values (using the variable i): There are $\binom{k}{2i}$ choices for where the + and $-$ signs will go, there are $\binom{2i}{i}$ choices for which of these will be + signs, and there are 2^{k-2i} choices for completing the string (since there are exactly two other characters, 0 and 1).

Now we will show that there is a bijection τ between S and T , and hence $|S| = |T|$. Given a string s in S , define the set $\tau(s)$ in the following way: if the m th character of s is a 0 or a +, include $2m - 1$ in $\tau(s)$; if the m th character of s is a 1 or a $-$, include $2m$ in $\tau(s)$.

Now we must show that $|\tau(s)| = k$. Let a be the number of 1’s in s , b the number of 0’s, c the number of +’s, and d the number of $-$ ’s. The size of $\tau(s)$ is $a + b + 2c$, which is equal to $a + b + c + d$, since $c = d$ by assumption, and furthermore, this is equal to k (being the total number of characters in s); hence $|S| \geq |T|$.

The map τ is reversible; if $2m - 1$ and $2m$ are in T , then the m th character of $\tau^{-1}(T)$ must be a +; if $2m - 1$ is in T but $2m$ isn’t, then the m th character of $\tau^{-1}(T)$ is a 0, etc. Every set T produces an element of S , since for that string s (using the notation above),

$$\begin{aligned} a + b + 2d &= k, \quad \text{and} \\ a + b + 2c &= k, \end{aligned}$$

so $c = d$ and $\tau^{-1}(T)$ is in S . This proves that τ is a bijection between S and T , and (1.2), as desired.

(2) **Solution using A=B.** This problem can also be solved using Sister Celine's method in $A=B$, pp. 55–72 (following Example 4.1.1), and Maple 11. First, we define $F_{n,k} = \binom{n}{k} \binom{2k}{k}$ and $S_n = \sum_{k \geq 0} F_{n,k}$, and look for a recurrence involving $F_{n,k}$. This can be done using the Maple code below.

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F := (n,k) -> binomial(n,k)*binomial(2*k,k):
II := 2: JJ := 2:
f := [seq(seq(simplify(convert(F(n+i,k+j)/F(n,k),factorial)), j=0..JJ), i=0..II)]:
f2 := [seq(seq('F'(n+i,k+j), j=0..JJ), i=0..II)]:
g := numer(simplify(sum(A[i]*f[i], i=1..nops(f)))):
h := {seq(coeffs(g,k)[i]=0, i=1..nops([coeffs(g,k)])), A[6]=0, A[2]=1}:
s := solve(h, {seq(A[i], i=1..nops(f))}):
s2 := subs(s, sum(A[i]*f2[i], i=1..nops(f2))=0);

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(Actually, we could have set $JJ := 1$; however, there are no identities with $II = 1$ and $JJ = 1$. Also, the conditions $A[6] = 0$ and $A[2] = 1$ were used to choose one particular solution.) This produces the identity

$$4F_{n,k} + F_{n,k+1} - \frac{2(2n+3)}{n+1}F_{n+1,k} - \frac{2n+3}{n+1}F_{n+1,k+1} + \frac{n+2}{n+1}F_{n+2,k+1} = 0, \quad (2.1)$$

for all $n \geq 1$, which can be derived from the algebraic identity

$$4 + \frac{2(n-k)(2k+1)}{(k+1)^2} - \frac{2(2n+3)}{n+1} \cdot \frac{n+1}{n-k+1} - \frac{2n+3}{n+1} \cdot \frac{2(n+1)(2k+1)}{(k+1)^2} + \frac{n+2}{n+1} \cdot \frac{2(2k+1)(n+1)(n+2)}{(n-k+1)(k+1)^2} = 0$$

by multiplying through by $F_{n,k}$ and using the properties of binomial coefficients. Then (2.1) is summed over all integers k , and since the coefficients are k -free, we get the equation

$$\begin{aligned}
4S_n + S_n - \frac{2(2n+3)}{n+1}S_{n+1} - \frac{2n+3}{n+1}S_{n+1} + \frac{n+2}{n+1}S_{n+2} &= 0, \quad \text{or} \\
5S_n - \frac{3(2n+3)}{n+1}S_{n+1} + \frac{n+2}{n+1}S_{n+2} &= 0. \quad (2.2)
\end{aligned}$$

Now, we repeat the same process for $\tilde{F}_{n,k} = \binom{n}{2k} \binom{2k}{k} 3^{n-2k}$ and $\tilde{S}_n = \sum_{k \geq 0} \tilde{F}_{n,k}$, where h is now

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h := {seq(coeffs(g,k)[i]=0, i=1..nops([coeffs(g,k)])), A[8]=0, A[9]=1};

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and which produces the recurrence

$$-\frac{4(n+1)}{n+2}\tilde{F}_{n,k+1} + \frac{9(n+1)}{n+2}\tilde{F}_{n,k+2} - \frac{3(2n+3)}{n+2}\tilde{F}_{n+1,k+2} + \tilde{F}_{n+2,k+2} = 0,$$

which, after summing over all k and simplifying, produces

$$\frac{5(n+1)}{n+2}\tilde{S}_n - \frac{3(2n+3)}{n+2}\tilde{S}_{n+1} + \tilde{S}_{n+2} = 0, \quad (2.3)$$

which is trivially equivalent to (2.2) when $n \geq 1$.

Now, $S_1 = \tilde{S}_1$, $S_2 = \tilde{S}_2$, and the fact that (2.2) and (2.3) are equivalent together prove that $S_n = \tilde{S}_n$, for all $n \geq 1$, by mathematical induction.

Vaguely interesting side note. The problem relates to the issue of whether there is a formula for the number of strings of length n with characters $+$, $-$, and 0 , $1, \dots, c-1$, where the number of $+$'s and the number of $-$'s are the same. If $c = 0$, then this formula is simply $\binom{n}{n/2}$ if n is even (and 0 if n is odd). If $c = 1$, then there does not seem to be a closed formula. If $c = 2$, this formula is $\binom{2n}{n}$ by (1.2) above, and the original problem asks whether there is a closed formula for the case $c = 3$. Evidently not, but there is an identity involving another finite sum (the statement to be proved).

The parallel worth considering is the fact that there are closed formulas for $\sum_k \binom{n}{k}^c$ when $c = 0$ (namely $n + 1$), $c = 1$ (namely 2^n), and $c = 2$ (namely $\binom{2n}{n}$), but none for $c = 3$, as proved by Marko Petkovšek, Herbert Wilf and Doron Zeilberger in $A=B$.