11258. *Proposed by Manuel Kauers, Research Institute for Symbolic Computation, Johannes Kepler University.* Let F_n denote the nth Fibonacci number, and let i denote $\sqrt{-1}$. Prove that

$$\sum_{k=0}^{\infty} \frac{F_{3^k} - 2F_{1+3^k}}{F_{3^k} + iF_{2\cdot 3^k}} = i + \frac{1}{2}\Big(1 - \sqrt{5}\Big).$$

Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ: The notation $F(x)$ will denote F_x, especially when x not a simple expression. Let

$$a_k = \frac{F(3^k) - 2F(1+3^k)}{F(3^k) + iF(2\cdot 3^k)} = \frac{F(3^k)\Big(F(3^k) - 2F(3^k+1)\Big)}{F(3^k)^2 + F(2\cdot 3^k)^2} - \frac{F(2\cdot 3^k)\Big(F(3^k) - 2F(3^k+1)\Big)}{F(3^k)^2 + F(2\cdot 3^k)^2}\, i. \tag{1}$$

The result is a consequence of the claim that

$$\sum_{i=0}^{k} a_i = \frac{-F(3^{k+1}-1)}{F(3^{k+1})} + \frac{F(3^{k+1})-1}{F(3^{k+1})} i, \tag{2}$$

for all $k \geq 0$, since $\lim\limits_{k\to+\infty} \dfrac{-F(3^{k+1}-1)}{F(3^{k+1})} = \dfrac{1-\sqrt{5}}{2}$ and $\lim\limits_{k\to+\infty} \dfrac{F(3^{k+1})-1}{F(3^{k+1})} = 1$.

The proof of (2) will be by induction. Clearly, (2) is true when $k = 0$, and will follow from the claims that

$$\frac{F(3^k)\Big(F(3^k) - 2F(3^k+1)\Big)}{F(3^k)^2 + F(2\cdot 3^k)^2} = \text{Re } a_k = \frac{-F(3^{k+1}-1)}{F(3^{k+1})} + \frac{F(3^k-1)}{F(3^k)} \quad \text{and} \tag{3a}$$

$$\frac{-F(2\cdot 3^k)\Big(F(3^k) - 2F(3^k+1)\Big)}{F(3^k)^2 + F(2\cdot 3^k)^2} = \text{Im } a_k = \frac{F(3^{k+1})-1}{F(3^{k+1})} - \frac{F(3^k)-1}{F(3^k)}, \tag{3b}$$

for all $k \geq 1$.

Fortunately (3a) and (3b) are true in a more general sense; if n is a positive odd integer, then

$$\frac{F_n(F_n - 2F_{n+1})}{F_n^2 + F_{2n}^2} = \frac{-F_{3n-1}}{F_{3n}} + \frac{F_{n-1}}{F_n} \quad \text{and} \tag{4a}$$

$$\frac{-F_{2n}(F_n - 2F_{n+1})}{F_n^2 + F_{2n}^2} = \frac{F_{3n}-1}{F_{3n}} - \frac{F_n - 1}{F_n}. \tag{4b}$$

Now, (4a) and (4b) can be proved using Binet's Formula and a CAS like Maple, but they can also be proven by using the following identities*:

Theorem. *The following hold for all integers:*
(i) "The Negative Identity": $F_{-n} = (-1)^{n+1}F_n$;
(ii) "Double Identity": $F_{2n} = F_n(2F_{n+1} - F_n)$ [1];
(iii) Catalan's Identity: $F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$;
(iv) d'Ocagne's Identity: $F_mF_{n+1} - F_nF_{m+1} = (-1)^{F}_{m-n}$;
(v) "Johnson's Identity": $F_aF_b - F_cF_d = (-1)^r(F_{a-r}F_{b-r} - F_{c-r}F_{d-r})$, *if a, b, c, d, and r are integers such that $a + b = c + d$.* [2]

To prove (4a), note that, using Catalan's Identity (with $n \leftarrow 2n$, $r \leftarrow n$), and the fact that n is odd,

$$F_nF_{3n} = F_{2n}^2 - (-1)^n F_n^2 = F_{2n}^2 + F_n^2.$$

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* Taken from `http://mathworld.wolfram.com/FibonacciNumber.html`

Then, using the Double Identity,

$$-F_{2n} \cdot (F_{2n}^2 + F_n^2) = F_n(F_n - 2F_{n+1}) \cdot (F_n F_{3n});$$

also, using d'Ocagne's Identity (with $m \leftarrow n-1$, $n \leftarrow 3n-1$), the Negative Identity, and the fact that n is odd,

$$F_{n-1}F_{3n} - F_{3n-1}F_n = (-1)^{3n-1}F_{-2n} = (-1)^{3n-1}(-1)^{2n+1}F_{2n} = -F_{2n}.$$

The two equalities above show that

$$F_n(F_n - 2F_{n+1}) \cdot (F_n F_{3n}) = -F_{2n} \cdot (F_{2n}^2 + F_n^2) = (-F_{3n-1}F_n + F_{n-1}F_{3n}) \cdot (F_{2n}^2 + F_n^2),$$

and dividing both sides by $F_n F_{3n}(F_{2n}^2 + F_n^2)$ yields (4a).

To prove (4b), note that (using the defintion of F_n)

$$F_n - 2F_{n+1} = -F_{n+1} + (F_n - F_{n+1}) = -F_{n+1} - F_{n-1};$$

This implies that

$$\begin{aligned}-F_{2n}(F_n - 2F_{n+1}) &= F_{2n}F_{n+1} + F_{2n}F_{n-1} = -F_n + [F_{2n}F_{n+1} - F_{2n}(-1)^n F_{n-1} + F_n] \\ &= -F_n + [F_{2n}F_{n+1} + (-1)^n(F_{2n}F_{1-n} - F_nF_1)],\end{aligned}$$

by the Negative Identity, n being odd, and the fact that $F_1 = 1$,

$$= -F_n + F_{3n} \cdot F_1,$$

by Johnson's Identity, where $a = 3n$, $b = 1$, $c = 2n$, $d = n+1$, and $r = n$,

$$= F_n(F_{3n} - 1) - F_{3n}(F_n - 1).$$

Then

$$-F_{2n}(F_n - 2F_{n+1}) \cdot F_{3n}F_n = [F_n(F_{3n} - 1) - F_{3n}(F_n - 1)] \cdot (F_{2n}^2 + F_n^2),$$

and dividing this equation by $F_{3n}F_n(F_{2n}^2 + F_n^2)$ produces (4b).

What happens if the Lucas numbers* are used instead of the Fibonacci numbers?

It appears that

$$\sum_{k=0}^{\infty} \frac{L_{3^k} - 2L_{1+3^k}}{L_{3^k} + iL_{2\cdot 3^k}} = \frac{1}{2}\left(1 - \sqrt{5}\right) + Si,$$

where $S = 1 + 2\sum_{k=1}^{\infty} \frac{1}{F_{3^k}}$. [I approximated S and then started a thread in `sci.math` asking people to "identify the number." Posters whose responses eventually led to this formula are Raymond Manzoni, Clive Tooth ("The Last Danish Pastry"), and Robert Israel.]

Note: While trying to establish a value for S, I came upon Manuel Kauers's paper on the SumCracker package [3], which he used to discover the identity

$$\sum_{k=0}^{n} \frac{F_{3^k} - 2F_{1+3^k}}{F_{3^k} + iF_{2\cdot 3^k}} = \frac{(2+i)F_{3^n} - (1+i)F_{3^n+1} - iF_{2\cdot 3^n} - F_{2\cdot 3^n+1}}{F_{3^n} - F_{3^n+1} + iF_{2\cdot 3^n+1}},$$

which provides a quicker proof.

References

[1] A. Brousseau, "Fibonacci Numbers and Geometry." *Fib. Quart.* **10**, 303–318, 1972.

[2] B. Johnson, "Fibonacci Identities by Matrix Methods and Generalisation to Related Sequences." March 25, 2003.)

[3] Manuel Kauers, "SumCracker: A Package for Manipulating Symbolic Sums and Related Objects", submitted to Elsevier Science.

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* The Lucas numbers satisfy the Fibonacci relation $L_n = L_{n-1} + L_{n-2}$, but start off differently: $L_1 = 1$ and $L_2 = 3$.