11218. *Proposed by Gary Gordon, Lafayette College, Easton, PA.* Consider the following algorithm, which takes as input a positive integer n and proceeds by *rounds*, listing in each round certain positive integers between 1 and n inclusive, ultimately producing as output a positive integer $f(n)$, the last number to be listed. In the 0th round, list 1. In the first round, list, in increasing order, all primes less than n. In the second round, list in increasing order all numbers that have not yet been listed and are of the form $2p$, where p is prime. Continue in this fashion, listing numbers of the form $3p$, $4p$, and so on until all numbers between 1 and n have been listed. Thus $f(10) = 8$ because the list eventually reaches the state $\{1, 2, 3, 5, 7, 4, 6, 10, 9, 8\}$, while $f(20) = 16$ and $f(30) = 27$.
(a) Find $f(2006)$.
(b) Describe the range of f.
(c) Find $\liminf_{n\to+\infty} \frac{f(n)}{n}$ and $\limsup_{n\to+\infty} \frac{f(n)}{n}$.

Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ: We proceed by establishing several intermediary results about the function f, using some extra functions. These extra functions have bizarre behavior when $n = 1$, so we will be concentrating on $n \geq 2$.

The first of these will be $R(n)$, which is the first round in which n appears; then $r(n)$ will be defined to be $R(f(n))$. Note that if $n \geq 2$ appears in round m, then $n = m \cdot p$, where p is a prime. To find the first round in which n appears, we need to make p be as large as possible and simultaneously remain a divisor of n. Thus, to minimize m, we need to let p be the *greatest prime factor* of n, which will be denoted by $\mathrm{GPF}(n)$. Then we will have $R(n) = \frac{n}{\mathrm{GPF}(n)}$, for $n \geq 2$. Note that $f(n) = R(k) \cdot \mathrm{GPF}(k)$, where $k = f(n)$.

Note that the last number in $\{1, \ldots, n\}$ to be listed will be one which maximizes $R(k)$ among the integers k between 1 and n; thus $r(n) = \max_{2\leq k\leq n} R(k)$, when $n \geq 2$. Now we present some "obvious" facts about f and r:

Lemma 1. *If $n \geq 1$ is an integer, then:*
(a) *f is non-decreasing;*
(b) *r is non-decreasing;*
(c) *$f(n) \leq n$;*
(d) *n is in the range of f iff $n = f(n)$;*
(e) *$f(n) = m$ iff for all k between 1 and n: $R(m) \geq R(k)$, and if $R(m) = R(k)$, then $m \geq k$;*
(f) *n is in the range of f iff $R(n) \geq R(k)$ for all k between 1 and n;*
(g) *If $n \geq 2$, $R(2n) = 2R(n)$.*

To prove Lemma 1(d), note that the "if" part is trivial; now suppose $n = f(k)$ for some $k \geq 1$. If $k > n$, then the elements in $\{1, \ldots, n-1\} \cup \{n+1, \ldots, k\}$ are listed before n in the algorithm described above. But then just the elements $\{1, \ldots, n-1\}$ are all listed before n, which implies that $f(n) = n$, which is what we wanted to show.

Part (e) is by definition, and part (f) follows from parts (d) and (e).

Part (g) follows since $\mathrm{GPF}(n) = \mathrm{GPF}(2n)$, when $n \geq 2$.

Now we move on to some non-obvious properties of f:

Lemma 2. *If $n \geq 2$ is in the range of f, then $\mathrm{GPF}(n) \leq 3$. Consequently, $n = 2^a \cdot 3^b$, for some nonnegative integers a, b.*

Proof: Suppose $n \geq 2$ is in the range of f. Choose m to be the largest power of 2 less than n, so that $m < n$ and $n \leq 2m$. Then

$$\frac{n}{4} \leq \frac{m}{2} = R(m) \leq r(m) \leq r(n) = R(n) = \frac{n}{\mathrm{GPF}(n)},$$

due to the form of m, the definition of $r(m)$, Lemma 1(b), the assumption that $f(n) = n$, and the definition of R. But this implies $\mathrm{GPF}(n) \leq 4$, which is equivalent to what we wanted to prove. This proves Lemma 2. ■

However, not every number of the form $2^a \cdot 3^b$ is in the range of f; in particular, $R(9) = 3 < 4 = R(8)$, so by Lemma 1(f), 9 is not in the range of f. The question of whether $f(2^a \cdot 3^b) = 2^a \cdot 3^b$ will be answered in stages, starting with:

Lemma 3. *n is in the range of f iff $2n$ is in the range of f.*

Proof: Lemma 3 is true if $n = 1$, so we assume that $n \geq 2$. We start by proving the "if" part: If $2n$ is in the range of f, then Lemma 1(f) implies

$$R(k) \leq R(2n), \quad \forall k \in \{2, \ldots, 2n\}. \tag{$\star$}$$

Now choose an i between 2 and n. Then, from $(\star)$ and Lemma 1(g), we deduce

$$2R(i) = R(2i) \leq R(2n) = 2R(n).$$

This inequality implies $R(i) \leq R(n)$ whenever $2 \leq i \leq n$. Thus n is in the range of f, by Lemma 1(f).

Now for the "only if" part. We assume that $R(k) \leq R(n)$, for all $k \in \{2, \ldots, n\}$, and need to show $(\star)$. Thus, let $i \in \{2, \ldots, 2n\}$.

If i is not a power of 2, then $\mathrm{GPF}\,(i) \geq 3$, and

$$R(i) = \frac{i}{\mathrm{GPF}\,(i)} \leq \frac{2n}{\mathrm{GPF}\,(i)} \leq \frac{2n}{3} \leq \frac{2n}{\mathrm{GPF}\,(n)} = \frac{2n}{\mathrm{GPF}\,(2n)} = R(n).$$

If $i = 2$, $R(i) = 1 \leq R(n)$, as $n > 1$. Lastly, if i is any other power of 2 between 2 and $2n$,

$$R(i) = 2R\left(\frac{i}{2}\right) \leq 2R(n) = R(2n),$$

which is the last part needed to show $(\star)$. Consequently, $2n$ is in the range of f, finishing the proof of Lemma 3. ∎

Before determining which values of k have $f(3^k) = 3^k$, we remark that Lemma 2 states that we only need to examine numbers of the form $2^a \cdot 3^b$ to determine $f(n)$; in fact, $f(n)$ will be the maximum f value of these numbers. Lemma 3 states even more; we only need to consider numbers of the form $2^a \cdot 3^b$ where $2^{a+1} \cdot 3^b > n$, since any number of the form $2^c \cdot 3^b$ will be eliminated before $2^a \cdot 3^b$ if $c < a$. Then we only need to determine what order these numbers are eliminated, and choose the one eliminated last. We have thus shown the following, where $L(i) = \left\lfloor \lg \frac{n}{3^i} \right\rfloor = \lfloor \lg n - i \lg 3 \rfloor$, and $\lg n$ denotes the logarithm of n base 2:

Lemma 4. *The following algorithm calculates $f(n)$, for $n \geq 2$; furthermore, this algorithm is polynomial time in $\lg n$, and also the number of bits necessary to express n.*

(a) *Calculate $a_0 = 2^{L(0)-1}$ and $b_0 = 2^{L(0)}$;*
(b) *Calculate $a_i = 2^{L(i)} \cdot 3^{i-1}$ and $b_i = 2^{L(i)} \cdot 3^i$, for $i = 1 \ldots, \lfloor \log_3 n \rfloor$;*
(c) *Determine the value j between 0 and $\lfloor \log_3 n \rfloor$ such that $a_j \geq a_i$, and if equality holds, $b_j > b_i$;*
(d) *Return b_j.*

Now we can determine when $f(3^k) = 3^k$:

Lemma 5. *$f(3^k) = 3^k$ iff $k \leq 1$ or $(k-1)\lg 3 > \lfloor k \lg 3 \rfloor - 1$.*

Proof: The result holds if $k \leq 1$, so suppose $k \geq 2$. Using the notation of Lemma 4, we want to show that $a_k > a_i$ for all $0 \leq i < k$.

First, we will concentrate on the case where $i > 0$; $a_k > a_i$ iff the following inequalities also hold, which are all equivalent by the properties of logarithms:

$$2^{\lfloor \lg 3^k - k \lg 3 \rfloor} \cdot 3^{k-1} > 2^{\lfloor \lg 3^k - i \lg 3 \rfloor} \cdot 3^{i-1}$$
$$2^{\lfloor (k-k) \lg 3 \rfloor} \cdot 3^k > 2^{\lfloor (k-i) \lg 3 \rfloor} \cdot 3^i$$
$$3^{k-i} > 2^{\lfloor (k-i) \lg 3 \rfloor}$$
$$(k-i) \lg 3 > \lfloor (k-i) \lg 3 \rfloor$$

Since the last inequality is true when $k-i$ is replaced by an arbitrary positive integer (and $\lg 3$ is irrational), this proves the intermediate claim.

Now we can state: $f(3^k) = 3^k$ iff $a_k > a_0$. (Note that we cannot have $a_k = a_0$, because 3 divides a_k but not a_0.) But the inequality $a_k > a_0$ is equivalent to the inequality

$$3^{k-1} > 2^{\lfloor k \lg 3 \rfloor - 1},$$

which, after some algebra, is equivalent to $(k-1)\lg 3 > \lfloor k \lg 3 \rfloor - 1$. ■

Now we turn to the questions which were asked.

(a) $f(2006) = 1944 = 2^3 \cdot 3^5$. Note that 1944 is the largest integer ≤ 2006 of the form $2^a \cdot 3^b$, and that $f(3^5) = 3^5$ by Lemma 5. (This author wonders whether the year 1944 has any significance for the proposer.)

[To show how much better the algorithm in Lemma 4 is than the original, Maple was used on a Intel Celeron processor (running at 1.4 GHz) to determine that $f(10^{100}) = 2^{332}$. The calculation took two seconds and 4.25M of memory.]

(b) Combining Lemmas 3 and 5 implies that the range of f is

$$\left\{ 2^a \cdot 3^b : a, b \geq 0, \quad a, b \in Z, \quad \text{and} \quad \left(b \leq 1 \quad \text{or} \quad (k-1)\lg 3 > \lfloor k \lg 3 \rfloor - 1 \right) \right\}.$$

(c) Lemma 1(c) implies that $\limsup_{n\to+\infty} \frac{f(n)}{n} \leq 1$, and since there are an infinite number of integers n such that $\frac{f(n)}{n} = 1$, $\limsup_{n\to+\infty} \frac{f(n)}{n} = 1$.

We proceed in a similar way to find $\liminf_{n\to+\infty} \frac{f(n)}{n}$. First of all, Lemma 2(a), Lemma 3, and the fact that $f(3) = 3$ imply that $f(2^k) = 2^k$ and $f(3 \cdot 2^k) = 3 \cdot 2^k$.

If $2 \cdot 2^k \leq n \leq 3 \cdot 2^k$ for some integer k, then (since f is non-decreasing) $\frac{f(n)}{n} \geq \frac{f(2^{k+1})}{3 \cdot 2^k} = \frac{2^{k+1}}{3 \cdot 2^k} = \frac{2}{3}$, and if $3 \cdot 2^k \leq n \leq 4 \cdot 2^k$, $\frac{f(n)}{n} \geq \frac{f(3 \cdot 2^k)}{4 \cdot 2^k} = \frac{3}{4}$. Thus, since $\frac{f(n)}{n} \geq \frac{2}{3}$ for all $n \geq 2$, $\liminf_{n\to+\infty} \frac{f(n)}{n} \geq \frac{2}{3}$.

Now we consider the (strictly) increasing sequence of positive integers $3 \cdot 2^k - 1$. If we can show that $f(3 \cdot 2^k - 1) = 2 \cdot 2^k$, then we can deduce

$$\liminf_{n\to+\infty} \frac{f(n)}{n} \leq \lim_{k\to+\infty} \frac{2 \cdot 2^k}{3 \cdot 2^k - 1} = \frac{2}{3},$$

and we will find that $\liminf_{n\to+\infty} \frac{f(n)}{n} = \frac{2}{3}$.

To determine $f(3 \cdot 2^k - 1)$, note that $r(2 \cdot 2^k) = 2^k = r(3 \cdot 2^k)$. Since r is nondecreasing (by Lemma 1(b)), $r(3 \cdot 2^k - 1) = 2^k$ as well. Then (if we define $p(n)$ to be GPF $(f(n))$):

$$2^k p(3 \cdot 2^k - 1) = f(3 \cdot 2^k - 1) \geq f(2 \cdot 2^k) = 2 \cdot 2^k, \quad \text{and}$$
$$2^k p(3 \cdot 2^k - 1) = f(3 \cdot 2^k - 1) \leq 3 \cdot 2^k - 1 < 3 \cdot 2^k,$$

which together imply $2 \leq p(3 \cdot 2^k - 1) < 3$, so $p(3 \cdot 2^k - 1) = 2$ and $f(3 \cdot 2^k - 1) = p(3 \cdot 2^k - 1) \cdot r(3 \cdot 2^k - 1) = 2 \cdot 2^k$, as desired.