11208. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL.* (Abbreviated) Let M_n be the stage-n Menger sponge, where M_0 is the unit cube. The *chromatic number* of a surface is the minimum number of colors that suffice to color any map drawn on that surface. Find the chromatic number of M_n.

Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ:

Theorem 1. *The chromatic number of M_n is*

$$\left\lfloor \frac{7}{2} + \frac{\sqrt{134064 \cdot 20^n + 242592 \cdot 8^n - 358967}}{266} \right\rfloor.$$

This solution requires the use of a published result:

Theorem 2. (Ringel, Youngs) *The chromatic number of the orientable surface of genus g is* $\left\lfloor \dfrac{7+\sqrt{48g+1}}{2} \right\rfloor$.

Proof. G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci.* U.S.A. **60** (1968), 438–445. □

Since the Euler characteristic of an orientable surface of genus g is $2-2g$, it suffices to find the Euler characteristic of M_n (which will be denoted μ_n; the Euler characteristic of an arbitrary set A will be denoted $\chi(A)$). We will need another result, about the Euler characteristic of the Sierpinski Gasket.

Definition. *Let the stage-n Sierpinski Gasket be S_n, defined as follows: S_0 is defined to be the unit square, and given S_{n-1}, construct S_n by drilling each of the 8^{n-1} sub-squares of edge-length 3^{1-n} in S_{n-1}. The Euler characteristic of S_n will be denoted γ_n.*

Lemma 3. *The Euler characteristic of S_n is* $\gamma_n = \dfrac{16}{7} - \dfrac{2}{7} \cdot 8^n$.

Proof of Lemma 3. The genus of S_n is equal to the number of "holes" in S_n, which is $1+8+8^2+\cdots+8^{n-1} = \dfrac{8^n-1}{7}$. The Euler characteristic of S_n is thus $2 - 2\cdot\dfrac{8^n-1}{7} = \dfrac{16}{7} - \dfrac{2}{7}\cdot 8^n$. □

Lemma 4. *The Euler characteristic of M_n, μ_n, satisfies the recurrence:*

$$\begin{aligned} \mu_0 &= 2, \\ \mu_{n+1} &= 20\mu_n - \frac{384}{7} + \frac{48}{7}\cdot 8^n, \quad n \geq 0. \end{aligned} \tag{$\star$}$$

Proof of Lemma 4. Clearly $\mu_0 = 2$. Now suppose $n \geq 1$.

We will view M_{n+1} as 20 copies of M_n, consisting of eight "corner cubes" (each adjacent to three other copies of M_n) and twelve "edge cubes" (each adjacent to two other copies of M_n). Fix n, and let N_0 consist of the eight corner cubes of M_{n+1}, and for each k between 1 and 12, let N_k be the union of N_{k-1} and one of the edge cubes not in N_k. (It does not matter which one in particular.) Thus $N_{12} = M_{n+1}$.

Since N_0 is the union of eight pairwise disjoint copies of M_n, the Euler characteristic of N_0 is $8\cdot\mu_n$. Now pick a k between 1 and 12. N_k is the union of N_{k-1} and a copy of M_n. The Euler characteristic satisfies the following inclusion-exclusion equality:

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Since the intersection of N_{k-1} and the new copy of M_n is two disjoint copies of S_n,

$$\chi(N_k) = \chi(N_{k-1} \cup M_n) = \chi(N_{k-1}) + \chi(M_n) - \chi(N_{k-1} \cap M_n) = \chi(N_{k-1}) + \mu_n - \chi(2S_n) = \chi(N_{k-1}) + \mu_n - 2\gamma_n.$$

Repeated iteration of this equality, with an application of Lemma 3, yields

$$\mu_{n+1} = \chi(M_{n+1}) = \chi(N_{12}) = \chi(N_0) + 12\mu_n - 24\gamma_n = 8\mu_n + 12\mu_n - 24\cdot\left(\frac{16}{7} - \frac{2}{7}\cdot 8^n\right),$$

which is equivalent to what we wanted to show. ☐

Now we can prove Theorem 1. First, we solve the recurrence given by Lemma 4. This is a linear non-homogeneous recurrence with constant coefficients, and it can be solved using methods found in any introductory combinatorics book, such as Rosen's *Discrete Mathematics and Its Applications*.

Briefly, we first solve the homogeneous recurrence $h_{n+1} = 20h_n$ (the recurrence $(\star)$, without the terms not involving μ); it is easy to see that a solution is of the form $h_n = C_1 \cdot 20^n$. Then we look for one particular solution to the recurrence

$$p_{n+1} = 20p_n - \frac{384}{7} + \frac{48}{7} \cdot 8^n,$$

by looking for a solution of the form

$$p_n = C_2 \cdot 1^n + C_3 \cdot 8^n.$$

It turns out that $C_2 = \dfrac{384}{133}$ and $C_3 = -\dfrac{4}{7}$ makes p_n satisfy $(\star)$, so

$$p_n = \frac{384}{133} - \frac{4}{7} \cdot 8^n.$$

Then any solution to $(\star)$ is of the form

$$\mu_n = h_n + p_n = C_1 \cdot 20^n + \frac{384}{133} - \frac{4}{7} \cdot 8^n,$$

and all we need to do is to choose C_1 so that $\mu_0 = 2$. To have this, we must have $C_1 = -\dfrac{6}{19}$, so

$$\mu_n = -\frac{6}{19} \cdot 20^n + \frac{384}{133} - \frac{4}{7} \cdot 8^n.$$

Now we substitute $g = \dfrac{2 - \mu_n}{2}$ into the formula given in Theorem 2, and obtain Theorem 1 after simplifying. ☐

The solver proposes the following conjecture:

Conjecture. *This problem was inspired by the confusion between the notations for chromatic number and Euler characteristic.*