

Series:

- $\sum_{i=0}^{\infty} a_i = a_0 + a_1 + a_2 + \dots$

You need to be able to recognize a pattern in a series and write it using \sum notation, write out the first few terms of a series given by \sum notation,

- Geometric series:

$\sum_{i=0}^{\infty} aq^i = a + aq + aq^2 + aq^3 + \dots$ is convergent if the $|q| < 1$, divergent if $|q| \geq 1$. If it is convergent then the sum is $\frac{a}{1-q}$. You need to be able to recognize if a series is geometric or not, and find the sum of the series.

- Ratio test:

Let $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = c$. If $c < 1$ then $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. If $c > 1$ then the series is divergent. If $c = 1$ then the test is inconclusive. For example $\sum_{i=1}^{\infty} \frac{2^i}{(i+1)!}$ is convergent. The ratio test is inconclusive for $\sum_{i=1}^{\infty} \frac{2i}{3i+2}$.

- Root test:

Let $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = c$. If $c < 1$ then $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. If $c > 1$ then the series is divergent. If $c = 1$ then the test is inconclusive. For example $\sum_{n=1}^{\infty} \left(\frac{n-2}{2n+3}\right)^n$ is convergent.

- Alternating series:

$\sum_{i=0}^{\infty} (-1)^i a_i$ is convergent if a_i is decreasing and $\lim_{i \rightarrow \infty} a_i = 0$. For example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent.

- Integral test:

If f is a positive decreasing function and $a_n = f(n)$, then either both $\int_1^{\infty} f(x) dx$ and $\sum_{n=1}^{\infty} a_n$ are convergent or both are divergent. For example $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent since $\int_1^{\infty} \frac{1}{x} dx$ is divergent.

- Comparison test:

Let $0 \leq a_i \leq b_i$. If $\sum_{i=1}^{\infty} b_i$ is convergent then so is $\sum_{i=1}^{\infty} a_i$. If $\sum_{i=1}^{\infty} a_i$ is divergent then so is $\sum_{i=1}^{\infty} b_i$. For example $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is convergent since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is.

- Limit comparison test:

If a_n and b_n are positive $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ where c is finite, then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent or both diverge. For example $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

- n -th term test:

If $\sum_{i=1}^{\infty} a_i$ is convergent then $\lim_{i \rightarrow \infty} a_i = 0$. If the limit is not 0 then the series is not convergent. For example $\sum_{n=1}^{\infty} \frac{2n}{3n+2}$ is divergent.

- Absolute convergence:

If the series $\sum_{i=0}^{\infty} |a_i|$ is convergent then the series $\sum_{i=0}^{\infty} a_i$ is called absolutely convergent. Every absolutely convergent series is convergent. For example $1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \dots$ is convergent because it's absolutely convergent. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent but not absolutely convergent.

- Taylor series:

Sometimes a series can be recognized as a Taylor series evaluated at a certain value. For example $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots = \cos(1)$.

Taylor series:

- The Taylor series of f around c is the power series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!} (x-c)^i = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!} (x-c)^2 + \frac{f'''(c)}{3!} (x-c)^3 + \dots$$

You need to be able to find and recognize the Taylor series of simple functions like \sin , \cos , e^x , $\frac{1}{x}$, $(1+x)^p$ using the definition.

- A Taylor series, like any power series, always has an interval of convergence with center at c . The distance between the center and the endpoints is called the radius of convergence. We can find the radius of convergence using the ratio test. The ratio test is inconclusive at the endpoints of this interval. So we need to consider the endpoints separately. The Taylor series of f may or may not converge to f but it usually does.

- A Taylor series can be differentiated and integrated term by term. The new series has the same radius of convergence. When you integrate you need special care to find the value of the constant term. For example you need to be able to find the Taylor series of $\ln(x)$, $\arctan(x)$ from the Taylor series of $\frac{1}{x}$ and $\frac{1}{1+x^2}$, or get the Taylor series of $\cos(x)$ from that of $\sin(x)$.

- We can substitute cx^k for x in a Taylor series or multiply a Taylor series by x^k . For example you can get the Taylor series of $\frac{x}{e^{x^2}}$ from the Taylor series of e^x by substituting $-x^2$ for x and then multiplying the terms by x .

- Given a Taylor series of f around c you need to be able to find the values of $f^{(i)}(c)$.

- The Taylor series can be used for finding limits of functions such as $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$ by replacing the function by its Taylor series.