

REVIEW

SECTIONS 10.1-10.2-10.3

Answer the following questions about Taylor series. If you are asked to find a Taylor series, you may start with a series that you already know and modify it or you may derive the series "from scratch". Also, if you are asked to find a Taylor series, either give the answer in summation notation, or give at least the first 4 non zero terms so that the pattern is apparent.

B1. a) Let $f(x) = e^{-\frac{x^2}{2}}$.

Find the Taylor series for $f(x)$ about $x = 0$.

- b) Find the Taylor series about $x = 0$ for the indefinite integral of $f(x)$ by integrating term by term the Taylor series you obtained above for $f(x)$.

Answer: a) $e^{-\frac{x^2}{2}} = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{2^i (i!)}$ b) $\int e^{-\frac{x^2}{2}} dx = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)2^i i!} + C$.

B2. There is no closed form antiderivative to the function $f(x) = \sin(x^2)$, but it is desired to find a numerical approximation to the following definite integral: $\int_0^1 \sin(x^2) dx$

- a) Find the Taylor series centered at $a = 0$ for $f(x) = \sin(x^2)$. Either express the series in summation notation, or show enough terms so that the pattern is apparent (at least three non-zero terms).
- b) Find the Taylor series centered at $a = 0$ for an antiderivative $F(x)$ of the function $f(x) = \sin(x^2)$. Do this by integrating the series from part a) term-by-term. Again, either express your answer in summation notation, or show enough terms so that the pattern is apparent.
- c) Using the series from part (b) for $F(x) = \int \sin(x^2) dx$ and the fundamental theorem of calculus, estimate the value of the following definite integral correct to three decimal places, i.e. correct to the thousandths place: $\int_0^1 \sin(x^2) dx$. In doing this, be sure to demonstrate how you know you have three decimal places of accuracy.

Answer: a) $\sin(x^2) \approx \sum_{i=0}^{\infty} \frac{(-1)^i x^{4i+2}}{(2i+1)!}$, b) $\int \sin(x^2) dx \approx \sum_{i=0}^{\infty} \frac{(-1)^i x^{4i+3}}{(4i+3)(2i+1)!} = \frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11} + \dots$ c) Approximate the integral by $P_7(1) - P_7(0) \approx .30952$. Approximate by $P_{11}(1) - P_{11}(0) \approx .310281$. The approximations agree to the first 3 decimal places, so the answer is .310.

B3. a) Find the Taylor series centered at $x = 0$ for $f(x) = -\ln(1 - 2x)$ by substituting into the series for $\ln(1 + x)$.

- b) Plot both $f(x)$ and its Taylor polynomial of various degrees and use the graph to guess what the interval of convergence is.

[Hint: Begin with the 3rd degree approximation. It's a good idea to use approximations as high as 10th degree!]

Answer: a) $2x + 2x^2 + \frac{8}{3}x^3 + 4x^4 + \dots$ b) $[-\frac{1}{2}, \frac{1}{2}]$.

B4. Use the fact that the Taylor series of $g(x) = \sin(x^2)$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

to find $g''(0)$, $g^{(3)}(0)$ and $g^{(10)}(0)$.

Answer: $g''(0) = 2$, $g^{(3)}(0) = 0$, $g^{(10)}(0) = 30240$.

B5. a) Write down the Taylor series for $\cos x$ at $x = 0$.

b) Use part a) to write down the Taylor series for $\cos \sqrt{x}$ at $x = 0$.

c) To what number does the series

$$1 - \frac{2}{2!} + \frac{4}{4!} - \frac{8}{6!} + \frac{16}{8!} - \dots$$

converge?

Answer: a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ b) $1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots$ c) the series converges to $\cos \sqrt{2}$.

B6. Compute the Taylor series for f around $x = 1$

1. $f(x) = 1/x$ 2. $f(x) = \frac{1}{2-x}$, 3. $f(x) = \ln(x)$

4. $f(x) = x^4$, 5. $f(x) = e^{-x}$, 6. $f(x) = e^{2x}$

Answer: 1. $\sum_{i=0}^{\infty} (-1)^i (x-1)^i$, 2. $\sum_{i=0}^{\infty} (x-1)^i$, 3. $\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (x-1)^i$, 4. $1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4$ 5. $e^{-1} \left(\sum_{i=0}^{\infty} \frac{1}{i!} (x-1)^i (-1)^i \right)$ 6. $e^2 \left(-1 + 2x + 2(x-1)^2 + \frac{4}{3}(x-1)^3 + \frac{2}{3}(x-1)^4 + \dots \right)$

B7. Write down the first 3 non zero terms of the Taylor series for $e^x \cos x$ around $x = 0$ from the series for e^x and $\cos x$.

Answer: $1 + x - \frac{x^3}{3}$

B8. a) Write down the Taylor series centered at 0 for $\frac{e^x - 1}{x}$, $\frac{\sin x}{x}$ and $\frac{1 - \cos x}{x^2}$, based on the series for e^x , $\sin x$ and $\cos x$.

b) At $x = 0$ the functions are 0/0. Evaluate the series at $x = 0$.

c) Check by l'Hopital's rule on $\frac{e^x - 1}{x}$, $\frac{\sin x}{x}$, and $\frac{1 - \cos x}{x^2}$.

Answer: a) $\frac{e^x - 1}{x} = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \dots$

$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \dots$, $\frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{1}{24}x^2 + \frac{1}{720}x^4 + \dots$ b) 1, 1, $\frac{1}{2}$.

B9. a) Write down the Taylor series for xe^x around $x = 0$.

b) Integrate and substitute $x = 1$ to find the sum of $\frac{1}{n!(n+2)}$.

Answer: a) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$, b) $F(x) = \int_0^x te^t dt = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$, $F(1) = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \int_0^1 te^t dt = 1$.

- B10. a) Find the Taylor series around $y = 0$ for $\frac{1}{1-y}$.
 b) Using your answer to part a), from $\sec x = \frac{1}{1 - (1 - \cos x)}$ find the Taylor series of $\sec x$ up to x^6 .

Answer: a) $1 + y + y^2 + y^3 + \dots$, b) $1 + \frac{x^2}{2!} + \frac{5}{24}x^4 + \dots$

- B11. Find a Taylor polynomial of degree at least four which is a solution to the initial value problem

$$\frac{dy}{dt} = t + y, \quad y(0) = 1$$

Answer: $y = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} \dots$

SECTION 10.4

- C1. A basket ball is dropped 10 feet and bounces back 6 feet.

After every fall it recovers $\frac{3}{5}$ of its height. What total distance does the ball travel, bouncing forever?

Answer: 40 feet.

- C2. A radioactive isotope is released into the air as an industrial by product. This isotope is not very stable due to radioactive decay. Two thirds of the original radioactive material loses its radioactivity after each month. If 10 grams of this isotope are released into the atmosphere at the end of the first and every subsequent month, then

- a) How much radioactive material is in the atmosphere at the end of the twelfth month? If the answer involves a sum, write it in closed form.
 b) In the long run, i.e. if the situation goes ad infinitum, what will be the amount of this radioactive isotope in the atmosphere at the end of each month?

Answer: a) $S_{12} = \frac{10 - (\frac{1}{3})^{12} \cdot 10}{1 - \frac{1}{3}}$, b) 15.

- C3. Find the exact sum of the infinite series

$$-1 + \frac{1}{3} + \frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \dots + \frac{4}{5^n} + \dots$$

Answer: $\frac{1}{3}$.

RATIO TEST

- D1. Use the ratio test to find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n} (x-1)^n$.

Answer: $\frac{1}{2} < x < \frac{3}{2}$.

D2. Consider the series $\sum_0^{\infty} \frac{5^n}{7^n} x^n$.

- a) Find the interval of convergence.
- b) Find the sum of the series in the interval of convergence.

Answer: a) $-\frac{7}{5} < x < \frac{7}{5}$. b) $\frac{7}{7-5x}$.

D3. Find the interval of convergence

1. $f(x) = \sum_0^{\infty} \left(\frac{x-1}{2}\right)^n$ 2. $f(x) = \sum_1^{\infty} \frac{2^n}{n^2 + 3n} (x-a)^{n-1}$
3. $f(x) = \sum_0^{\infty} \frac{1}{n+1} (x-a)^{n+1}$ 4. $f(x) = (x-2\pi) - \frac{(x-2\pi)^3}{3} + \frac{(x-2\pi)^5}{5} - \dots$

Answer: 1. $-1 < x < 3$, 2. $-\frac{1}{2} + a < x < \frac{1}{2} + a$, 3. $-1 + a < x < 1 + a$. 4. $2\pi - 1 < x < 1 + 2\pi$

D5. Use the ratio test to find the interval of convergence of

1. $\sum_{n=1}^{\infty} \frac{3^n}{n} (x+1)^{2n}$, 2. $\sum_{n=0}^{\infty} \frac{4^n}{n!} (x+3)^n$.

Answer: $-1 - \frac{1}{\sqrt{3}} < x < -1 + \frac{1}{\sqrt{3}}$, 2. every x .

SECTION 10.5

E1. Use the binomial series to compute with an error less than 1/1000:

1. $(1.5)^{1/4}$ 2. $(1.1)^{1.1}$

Answer: 1. $P_4(.5) \approx 1.10604$, error $\leq \max_{0 \leq x \leq .5} |f^{(5)}(x)| \frac{(.5)^5}{5!} \approx 0.00088$,

2. $P_1(.1) = 1.11$, error $\leq \max_{0 \leq x \leq .1} |f''(x)| \frac{(.1)^2}{2!} \approx 0.00055$,

E2. Use a Taylor polynomial of degree 2 to approximate $\ln(1.3)$.

- a) Which function do you use to derive your Taylor polynomial?
- b) Write P_2 for the function from a) and approximate $\ln(1.3)$.
- c) Find an upper bound on the error in your approximation.

Answer: a) $\ln(1+x)$ around $x = 0$, (or $\ln(x)$ around $x = 1$) b) $P_2 = x - \frac{x^2}{2}$, $P_2(.3) = .255$,
c) 0.009.

E3. a) Find the Taylor polynomial of degree 3 around $x = 0$ for the function $f(x) = \sqrt{1-x}$.

b) Use your answer to part a) to give approximate values to $\sqrt{0.6}$ and $\sqrt{0.8}$.

c) Which approximation in part b) is more accurate? Explain why.

Answer: a) $P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, b) $\sqrt{0.6} \approx 0.7760$, $\sqrt{0.8} \approx 0.8945$. c) more accurate for $\sqrt{0.8}$ because 0.2 is closer to 0 than 0.4 is.

APPENDIX C and D

F1. Write the complex numbers in polar form and perform the indicated calculations. Give your answer in polar coordinates.

1. $(1 + i)^{100}$, 2. $(3 + i\sqrt{3})^{24}$, 3. $(1 + i\sqrt{3})^{-\frac{1}{2}}$.

Answer: 1. -2^{50} , 2. 12^{12} , 3. $\frac{1}{2\sqrt{2}}(\sqrt{3} - i)$

F2. Let $z_1 = -3 - i\sqrt{3}$ and $z_2 = -1 + i\sqrt{3}$.

a) Find z_1z_2 and z_1/z_2 . Give your answer in cartesian form $z = x + iy$.

b) Put z_1 and z_2 into polar form, $z = re^{i\theta}$. Find z_1z_2 and z_1/z_2 using the polar form and verify that you get the same answer as in part a).

Answer: a) $z_1z_2 = 6 - 2i\sqrt{3}$, $z_1/z_2 = i\sqrt{3}$