What to do with the partial fractions?

After a partial fraction decomposition we have to face the following integrals:

1. $\displaystyle\int \frac{A}{(ax+b)^n}dx \quad$ when $n > 0$
2. $\displaystyle\int \frac{Ax+B}{(ax^2+bx+c)^n}dx \quad$ when $n > 0$ and ax^2+bx+c is irreducible
3. $\displaystyle\int \frac{A}{(ax^2+bx+c)^n}dx \quad$ when $n > 0$ and ax^2+bx+c is irreducible
4. $\displaystyle\int \frac{A}{(ax^2+b^2)^n}dx \quad$ when $n > 1$
5. $\displaystyle\int \frac{A}{ax^2+b^2}dx$

We can also assume that $a > 0$, since this can be achieved by factoring out -1 if necessary. Let us consider these cases one by one.

Case 1.

Substituting $w = ax + b$ helps. Then $dw = adx$ and so

$$\int \frac{A}{(ax+b)^n}dx = \frac{A}{a}\int \frac{1}{w^n}dw.$$

If $n \neq 1$ then this is equal to

$$\frac{A}{a}\frac{w^{-n+1}}{-n+1} + C = \frac{A}{a}\frac{(ax+b)^{-n+1}}{-n+1} + C.$$

If $n = 1$ then we have

$$\frac{A}{a}\ln|w| + C = \frac{A}{a}\ln|ax+b| + C.$$

Case 2.

We would like to get the derivative of ax^2+bx+c in the numerator, so that we can substitute $w = ax^2 + bx + c$.

$$\begin{aligned}\int \frac{Ax+B}{(ax^2+bx+c)^n}dx &= \frac{A}{2a}\int \frac{(2ax+b)+(2aB/A-b)}{(ax^2+bx+c)^n}dx \\ &= \frac{A}{2a}\int \frac{2ax+b}{(ax^2+bx+c)^n}dx + \frac{A}{2a}\int \frac{2aB/A-b}{(ax^2+bx+c)^n}dx.\end{aligned}$$

In Case 3. we are going to discuss what to do with the second integral. In the first integral the above mentioned substitution helps. We have $dw = (2ax + b)dx$ and so

$$\frac{A}{2a}\int \frac{2ax+b}{(ax^2+bx+c)^n}dx = \frac{A}{2a}\int \frac{dw}{w^n}$$

which we saw how to handle in Case 1. depending on n.

Case 3.

We complete the square in the denominator and substitute $w = x + \frac{b}{2a}$

$$\begin{aligned}\int \frac{A}{(ax^2+bx+c)^n}dx &= \int \frac{A}{\left(a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}\right)^n}dx \\ &= \int \frac{A}{\left(aw^2+c-\frac{b^2}{4a}\right)^n}dw.\end{aligned}$$

Since ax^2+bx+c is irreducible, the discriminant b^2-4ac is negative and so $c-\frac{b^2}{4a} > 0$ (note that $a > 0$). Thus, $c-\frac{b^2}{4a}$can be written as a square of a number and now we can proceed as in Case 4.

Case 4.

We use integration by parts to get a recursive formula. If

$$u = \frac{A}{(ax^2+b^2)^n} \quad \text{and} \quad v' = 1$$

then

$$u' = -\frac{2Anax}{(ax^2+b^2)^{n+1}} \quad \text{and} \quad v = x$$

and so

$$\begin{aligned}\int \frac{A}{(ax^2+b^2)^n}dx &= \frac{Ax}{(ax^2+b^2)^n} + 2An\int \frac{(ax^2+b^2)-b^2}{(ax^2+b^2)^{n+1}}dx \\ &= \frac{Ax}{(ax^2+b^2)^n} + 2An\int \frac{1}{(ax^2+b^2)^n}dx - 2Anb^2\int \frac{1}{(ax^2+b^2)^{n+1}}dx.\end{aligned}$$

We solve for the last term

$$\int \frac{A}{(ax^2+b^2)^{n+1}}dx = \frac{1}{2nb^2}\frac{Ax}{(ax^2+b^2)^n} + \frac{2n-1}{2nb^2}\int \frac{A}{(ax^2+b^2)^n}dx. \tag{1}$$

After applying this formula possibly several times, the exponent in the denominator becomes 1 and we can proceed as in Case 5.

Case 5.

We want to use the fact that $\arctan'(x) = \frac{1}{1+x^2}$. After some trickery we substitute $w = \frac{\sqrt{a}x}{b}$

$$\int \frac{A}{ax^2+b^2}dx = \int \frac{\frac{A}{b^2}}{\frac{a}{b^2}x^2+1}dx$$

$$\begin{aligned}
&= \frac{A}{b^2}\int \frac{1}{1+\left(\frac{\sqrt{a}x}{b}\right)^2}dx \\
&= \frac{A}{b^2}\int \frac{1}{1+w^2}\frac{b}{\sqrt{a}}dx \\
&= \frac{A}{b\sqrt{a}}\arctan(\frac{\sqrt{a}x}{b})+C
\end{aligned}$$

Put it together

Let's find

$$I=\int \frac{32x^7-176x^6+340x^5+96x^4-1545x^3+2742x^2-2148x+633}{(3x-5)(2x-3)^2(2x^2-x+1)^2}dx.$$

After finding the partial fraction decomposition . We need to find the following three integrals

$$I=\int \frac{1}{3x-5}dx+\int \frac{3}{(2x-3)^3}dx-\int \frac{4x-5}{(2x^2-x+1)^2}dx$$

a)

The first one fits Case 1. Substitute $w=3x-5$

$$\begin{aligned}
\int \frac{1}{3x-5}dx &= \frac{1}{3}\int \frac{1}{w}dw \\
&= \frac{1}{3}\ln|3x-5|+C_1.
\end{aligned}$$

b)

For the second we are again in Case 1. Substitute $w=2x-3$

$$\begin{aligned}
\int \frac{3}{(2x-3)^3}dx &= \frac{3}{2}\int \frac{1}{w^3}dw \\
&= \frac{3}{2}\frac{1}{-2w^2}+C_2 \\
&= -\frac{3}{4}\frac{1}{(2x-3)^2}+C_2.
\end{aligned}$$

c)

The third integral requires Case 2. After some algebra and the substitution $y=2x^2-x+1$we have

$$-\int \frac{4x-5}{(2x^2-x+1)^2}dx = -\int \frac{(4x-1)-4}{(2x^2-x+1)^2}dx$$

$$
\begin{aligned}
&= -\int \frac{4x-1}{(2x^2-x+1)^2}dx + \int \frac{4}{(2x^2-x+1)^2}dx \\
&= -\int \frac{1}{y^2}dy + \int \frac{4}{(2x^2-x+1)^2}dx \\
&= \frac{1}{2x^2-x+1} + C_3 + \\
&= \qquad + \int \frac{4}{(2x^2-x+1)^2}dx.
\end{aligned}
$$

For the last integral we complete the square, substitute $w = x - \frac{1}{4}$, apply equation 1 in Case 4 with $n = 1$ and apply our formula in Case 5:

$$
\begin{aligned}
\int \frac{4}{(2x^2-x+1)^2}dx &= \int \frac{4}{(2(x-\frac{1}{4})^2+1-\frac{1}{8})^2}dx \\
&= \int \frac{4}{(2w^2+\frac{7}{8})^2}dw \\
&= \frac{1}{2\cdot 1\cdot \frac{7}{8}} \cdot \frac{4w}{2w^2+\frac{7}{8}} + \frac{2\cdot 1-1}{2\cdot 1\cdot \frac{7}{8}} \int \frac{4}{2w^2+\frac{7}{8}}dw \qquad \text{by Case 4} \\
&= \frac{4}{7} \cdot \frac{4x-1}{2x^2-x+1} + \frac{4}{7} \cdot \frac{4\sqrt{8}}{\sqrt{7}\sqrt{2}} \arctan\left(\frac{\sqrt{2}\sqrt{8}w}{\sqrt{7}}\right) + C_4 \qquad \text{by case 5} \\
&= \frac{4}{7} \cdot \frac{4x-1}{2x^2-x+1} + \frac{32\sqrt{7}}{49} \arctan\left(\frac{4x-1}{\sqrt{7}}\right) + C_4.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I &= \frac{1}{3}\ln|3x-5| - \frac{3}{4}\frac{1}{(2x-3)^2} + \frac{1}{2x^2-x+1} + \frac{4}{7} \cdot \frac{4x-1}{2x^2-x+1} + \frac{32\sqrt{7}}{49} \arctan\left(\frac{4x-1}{\sqrt{7}}\right) + C \\
&= \frac{1}{3}\ln|3x-5| - \frac{3}{4}\frac{1}{(2x-3)^2} + \frac{1}{7}\frac{16x+3}{2x^2-x+1} + \frac{32\sqrt{7}}{49} \arctan\left(\frac{4x-1}{\sqrt{7}}\right) + C.
\end{aligned}
$$

Exercises:

Find the following integrals:

1. $\int \frac{3}{(-2x+1)^3}dx$
2. $\int_1^2 \frac{-2}{3x+1}dx$
3. $\int \frac{2}{4x^2+9}dx$
4. $\int \frac{1}{(x^2+4)^2}dx$
5. $\int \frac{2x-1}{x^2+2x+4}dx$
6. $\int \frac{1}{4x^2-9}dx$