Course: MAT 425
Numerical Analysis Final Exam and
Mathematics Department Qualifier Exam
Spring 2006

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Circle the type of exam: **Final** **Qualifier** **Both**

Circle name of instructor: **Gelb/Renaut** **Lopez**

1. (i) Derive the composite trapezoidal quadrature rule used to approximate $\int_a^b f(x)\mathrm{d}x$ when the integration interval is subdivided into M subdivisions of width $h = (b-a)/M$.

   (ii) Use the errors in approximating $f(x)$ by its Lagrange polynomial in each subinterval to estimate the error in the composite trapezoidal rule.

2. Consider the boundary value problem for $x \in [0, 1]$:

$$y'' = 4(y - x),\ y(0) = 0,\ y(1) = 2.$$

(a) Show that the BVP has a unique solution.

(b) Use the linear shooting method to solve the BVP.

3. Consider the finite-difference approximation of the first derivative of a smooth function $f(x)$:

$$f'(x_i) = \left(f(x_{i+1}) - f(x_{i-1})\right)/2h + E(f,h),$$

where $h = x_{i+1} - x_i$ and the error $E(f,h) = E_r(f,h) + E_t(f,h)$ has contributions from round-off and trunction errors.

Derive expressions for the round-off error $E_r(f,h)$ and the trunction error $E_t(f,h)$.

Use these to find an optimal value of h which minimizes $E = E_r + E_t$.

4. Consider the PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ x \in [0,1],\ t > 0;\ U(0,t) = U(1,t) = 0,\ U(x,0) = g(x).$$

Use the method of lines to show that $V(t+k) = [\exp(k\mathbf{A})]\, V(t)$ is an approximation of order h^2 to the PDE at grid points $x_i = ih$, $i = 1 \to N-1$, where $h = 1/N$, and $\mathbf{A}$ is the matrix

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & . & . & & & \\ & & . & . & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \end{bmatrix},$$

with eigenvalues

$$\lambda_i = -\frac{4}{h^2}\sin^2\left(\frac{i\pi}{2N}\right),\ i = 1 \to N-1.$$

Approximate $\exp(k\mathbf{A})$ by $1/(1 - k\mathbf{A})$ and show that the resulting finite-difference scheme for the approximate solution of the PDE is L-stable.