Qualifying Exam
in
Linear Algebra and Field Theory

August, 2006

Answer as many questions as you can, but be sure to answer sufficiently many questions to comprise 120 points, with at least 30 points from each section. Detailed, completely correct answers to a smaller number of questions will obtain more credit than partial answers to a larger number of questions. This is especially important for earning a Ph.D. level pass. Write your answer to each problem on a separate sheet, placing the problem number in the *upper right* corner. Be sure to justify all your answers. Do not cite any result which reduces a proof to a triviality.

Field Theory

1. [5,5,7,5] Let p be a prime.
   a. In terms of p, how many monic irreducible degree-3 polynomials in $\mathbb{F}_p[X]$ are there? Prove that your answer is correct.
   b. Show for any prime p, there is always a field of cardinality p^3.
   c. Give explicitly the construction, including addition and multiplication tables, of a field with 8 elements.
   d. Suppose F is a field of cardinality 81. What are the subfields of F? Prove that your answer is correct.

2. [5,7,5,7]
   a. Define the characteristic of a field.
   b. Show that the characteristic of a field must be either zero or a prime.
   c. Show that if F is a field of characteristic $p > 0$ and $x, y \in F$, then $(x+y)^p = x^p + y^p$.
   d. Let F be a finite field with characteristic $p > 0$. Show that $\#F = p^n$ for some positive integer n.

3. [8,8,5,7] Let $K = \{a + b\sqrt[4]{2} + c\left(\sqrt[4]{2}\right)^2 + d\left(\sqrt[4]{2}\right)^3 : a, b, c, d \in \mathbb{Q}\}$.
   a. Show that K is a subfield of $\mathbb{R}$ which contains $\mathbb{Q}$ and find $[K : \mathbb{Q}]$. Justify your answer.
   b. Show that $K \cong \mathbb{Q}[X]/\langle X^4 - 2\rangle$ by giving the isomorphism explicitly.
   c. Show that K contains a subfield which is isomorphic to $\mathbb{Q}[X]/\langle X^2 - 2\rangle$.
   d. Does the polynomial $X^4 - 2$ split in K? If not, what is the splitting field of $X^4 - 2$ over K? Justify your response.

Linear Algebra

4. [5,8,7,5,7,7] Let F be a field and let V be an F-vector space. Let W_1 and W_2 be subspaces of V.
   a. Show that $W_1 + W_2$ is the smallest subspace of V which contains both W_1 and W_2.
   b. Suppose W_1 and W_2 are finite dimensional. Show:
   $$\dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2).$$
   c. Suppose $V = \mathrm{Mat}_{2\times 2}(F)$ and
   $$W_1 = \{\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right) \in V : a, b, c \in F\},\ W_2 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \in V : a \in F\}.$$
   Show that W_1 and W_2 are subspaces of V.
   d. For V, W_1 and W_2 as in part *c*, find $\dim_F(W_1 + W_2)$ by first finding $\dim_F(W_1)$, $\dim_F(W_2)$ and $\dim_F(W_1 \cap W_2)$. Justify your computations.
   e. For V, W_1 and W_2 as in part *c*, determine whether it is true that $V = W_1 \oplus W_2$. Prove that your answer is correct.
   f. Suppose $F = \mathbb{R}$ or $\mathbb{C}$ and V is a finite dimensional inner product space. Show
   $$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp.$$

5. [7,7,7,7,7] Prove or disprove:
   a. If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.
   b. If E is an elementary matrix, then $\det(E^t) = \det(E)$.
   c. If F is a field, the function det: $\mathrm{Mat}_{n\times n}(F) \longrightarrow \mathrm{Mat}_{n\times n}(F)$ is an F-linear transformation.
   d. If A and B are similar matrices, then $\det(A) = \det(B)$.
   e. If A is an orthogonal real matrix, then $\det(A) = 1$.

6. [8,7,7] Let F be a field and let $A \in \mathrm{Mat}_{n\times n}(F)$ have characteristic polynomial
   $$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_0.$$
   a. Prove that A is invertible if and only if $a_0 \neq 0$.
   b. Suppose $a_0 \neq 0$. Write a formula for A^{-1} in terms of n, non-negative powers of A, and $a_0, \ldots, a_{n-1}$. Explain why your formula is correct.
   c. Let $W = \mathrm{span}(I, A, A^2, A^3, \ldots)$. Show that $\dim_F(W) \leq n$.

7. [8,7,8,7] Let $T : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ be the linear operator given by
   $$T(a_1, \cdots, a_7) = (a_2 + a_3 + a_4 + a_5 + a_6 + a_7, a_4 + a_5 + a_6, a_6, 0, 0, 0, a_7).$$
   a. Find the eigenvalues of T and bases for their corresponding eigenspaces. Explain why T is not diagonalizable.
   b. Find bases for the generalized eigenspaces of T.
   c. Find the minimum polynomial of T and the Jordan canonical form of T.
   d. Find a Jordan canonical basis associated to the Jordan canonical form from part *c*.