REAL ANALYSIS QUALIFYING EXAM — PART 1

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INSTRUCTIONS. Write on only one side of each sheet and leave reasonable margins. Put your name on each page. You may keep the exam sheet itself.

You may use any part of a problem (solved or not) in the solution of any other problem, and in any *later* part of the same problem. You may ignore hints, but you must follow explicit instructions.

This exam has 5 problems, and all problems have equal credit.

1. Let $f : [0, \infty) \to \mathbb{R}$. Assume that $f(0) = 0$, and that f is continuous at 0. For each $n \in \mathbb{N}$ define $g_n : [0, \infty) \to \mathbb{R}$ by $g_n(x) = f(x/n)$.

(a) Find the pointwise limit of the sequence (g_n).

(b) Prove that if

$$\lim_{n\to\infty} \big(\sup_{x\geq 0} |g_n(x)|\big) = \sup_{x\geq 0}\big(\lim_{n\to\infty} |g_n(x)|\big),$$

then f is identically 0.

2. For each $n \in \mathbb{N}$ let $f_n : [0, 1] \to \mathbb{R}$ be continuous. Assume that for each $x \in [0, 1]$ the sequence $(f_n(x))$ is decreasing and converges to 0. Prove that the sequence (f_n) converges uniformly. *Hint*: Use continuity of each f_n and compactness of $[0, 1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that f is differentiable on $\mathbb{R} \setminus \{0\}$ and that $\lim_{x\to 0} f'(x)$ exists.

(a) Prove that f is differentiable at 0.

(b) Prove that f' is continuous at 0.

4. Let (X, d) be a metric space, and let (x_n) and (y_n) be two Cauchy sequences in X. Prove that the real sequence $\big(d(x_n, y_n)\big)_{n=1}^{\infty}$ converges (in $\mathbb{R}$).

5. Let $f\colon [0, \infty) \to \mathbb{R}$ be continuous, and assume that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\lim_{a\to\infty} \frac{1}{a} \int_0^a f(x)\,\mathrm{d}x = 0.$$