**Course: MAT 423**
**Numerical Analysis**
**Mathematics Department Qualifier Exam**
**Fall 2006**

**Student Name:** ________________

**Student Email:** ________________

Please write your name on the top of each page of your answer
Only use one side of each sheet of paper.
No books or handouts allowed

1. (a) Give Newtons method for solving the system of equations $F(\mathbf{x}) = 0$.

(b) Find the Jacobian for the system of nonlinear equations

$$\begin{aligned} f(x, y) &= xy - y^3 - 1 = 0 \\ g(x, y) &= x^2 y + y - 5 = 0 \end{aligned}$$

Evaluate the Jacobian at the point $(x, y) = (2, 3)$ and form its LU factorization.

(c) From the initial guess $(x, y) = (2, 3)$ carry out one step of Newton's method.To solve the linear systems of equations that arise use the LU factorization obtained in part (b).

(d) Set up the iteration and the systems that must be solved for the second step of Newton's method with reuse of the Jacobian from the first step.

continued problem 1

2. Let $A_n$ be the $2 \times 2$ matrix given by

$$A_n = \begin{pmatrix} 1 & 2 \\ 2 & 4 + 1/n^2 \end{pmatrix}$$

(a) Find $A_n^{-1}$ and the condition number of $A_n$. ( Use the one norm to calculate the condition number).

(b) Let $n = 100$. Use Gaussian elimination without pivoting to solve $A_{100} = b$ using 5 significant figures at all stages of the calculation when

$$b = (1, 2 - 1/n^2)^T$$

.

(c) Repeat part (b) using 2 significant figures in the calculation.

(d) Explain the answers in parts (b) and (c).

continued problem 2

3. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of $A$

(b) Factor $A$ into a product $QR$ where $Q$ has an orthonormal set of column vectors and $R$ is upper triangular.

(c) Solve the least squares probem $A\mathbf{x} = \mathbf{b}$.

continued problem 3

4. Recall the power iteration method for computing the largest eigenvalue (in amplitude) of an $N \times N$ matrix $A$:

**Power Iteration Algorithm:**

- Choose $\mathbf{x_0}$ as an arbitrary non-zero vector.
- for $k = 1, 2, \cdots$, compute $\mathbf{x_k} = \mathbf{A}\mathbf{x_{k-1}}$.

(a) If $A$ has a unique eigenvalue $\lambda_1$ where $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_N|$, show that the power iteration method converges to a multiple of the corresponding eigenvector $\mathbf{v_1}$.

(b) What can be said about the convergence of the method?

(c) Why must we assume that $|\lambda_1|$ is strictly greater than the moduli of all the other eigenvalues?

(d) Apply the power iteration three times to obtain the maximum eigenvalue (in amplitude) for:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

continued problem 4