Algebra Qualifying Exam **Groups and Rings**
August 2006 **MAT 444**

Name: ______________________________

Do as many problems as you can. Write your answers on separate paper. Make sure your name is on every sheet.

1. Suppose G is a group of order 385.
   (a) (20 points) Prove that G has a subgroup H of order 77.
   (b) (10 points) Prove that H is cyclic.
   (c) (10 points) Prove that H is a normal subgroup of G.
   (d) (20 points) Prove that G has exactly one subgroup of order 77.

2. (a) (20 points) Suppose n and m are positive integers. Determine the number of distinct group homomorphisms $\mathbf{Z}_m \to \mathbf{Z}_n$. Prove your answer.
   (b) (20 points) Suppose H and K are finite abelian groups. Prove that there is a non-trivial group homomorphism $H \to K$ iff there is a non-trivial group homomorphism $K \to H$.
   (c) (10 points) Find finite groups H and K such that there exists a non-trivial group homomorphism $H \to K$ but the only homomorphism $K \to H$ is the trivial one. Justify your claim.

3. Suppose G is a finite group and S is the set of subgroups of G.
   (a) (10 points) Prove that conjugation gives a group action of G acting on S.
   (b) (10 points) If $G = S_3$, what are the orbits of this action?
   (c) (20 points) Prove that if G is a non-abelian simple group and H is a maximal subgroup of index n, then H has exactly n conjugate subgroups (including itself).

4. Suppose that $\varphi\colon R \to S$ is a ring homomorphism.
   (a) (10 points) Prove that $\ker\varphi$ is an ideal of R.
   (b) (20 points) State and prove the first isomorphism theorem for rings (also known as the fundamental homomorphism theorem).
   (c) (10 points) Prove that if J is an ideal of S, then its inverse image $\varphi^{-1}(J)$ is an ideal of R which contains $\ker\varphi$.
   (d) (10 points) Disprove: if I is an ideal of R, then $\varphi(I)$ is an ideal of S.
   (e) (10 points) Prove that if φ is surjective and I is an ideal of R, then $\varphi(I)$ is an ideal of S.

5. In $\mathbf{Z}[i]$, let $I = \langle 7 - i, 15 - 10i \rangle = \{(7 - i)a + (15 - 10i)b \mid a, b \in \mathbf{Z}[i]\}$, i.e., the ideal generated by $7 - i$ and $15 - 10i$.

   (a) (8 points) Explain briefly why I must be principal.

   (b) (12 points) Find a generator for I. Use a method which is reasonable for computing the generator of any ideal in $\mathbf{Z}[i]$ generated by two given elements, and show the steps of your computation.

   (c) (20 points)In the ring $\mathbf{Z}[\sqrt{-3}]$, show that $J = \langle 1 + \sqrt{-3}, 2 \rangle$, the ideal generated by $1 + \sqrt{-3}$ and 2, is not principal.

6. Let R be a commutative ring with $1 \neq 0$. An ideal I of R is *primary* if I is a proper ideal such that $ab \in I$ implies $a \in I$ or $b^n \in I$ for some $n \geq 1$.

   (a) (10 points) Prove that every prime ideal is primary.

   (b) (10 points) Prove that if p is a prime element of a PID and $m > 1$, then $\langle p^m \rangle$ is a primary ideal but not a prime ideal.

   (c) (10 points) For an ideal I, let

   $$\mathrm{rad}(I) = \{r \in R \mid r^n \in I \text{ for some } n > 0\}$$

   Prove that if I is an ideal of R, then $\mathrm{rad}(I)$ is an ideal of R.

   (d) (20 points) Prove that if I is a primary ideal, then $\mathrm{rad}(I)$ is a prime ideal.