MAT 442 | **December 2007** | **Linear Algebra**
Final Exam | | **Qualifying Exam**

Write your answers on separate paper. Make sure your name is on every sheet. You may use any results from the course or text, except when asked to prove those results (which are marked as "result from course").

1. (10 points each) Let V be the vector space of $n \times n$ matrices over a field F.
   (a) Prove that the trace, $\mathrm{Tr}(A)$, gives a linear functional on V.
   (b) Let W be the set of matrices of trace 0. Prove that W is a subspace of V.
   (c) Determine the dimension of W, and prove your answer.

2. (10 points each) Suppose A is an $n \times n$ skew-symmetric matrix (i.e., $A^t = -A$) with entries from $\mathbf{R}$, and that A is not the zero matrix.
   (a) Prove that if n is odd, then 0 is an eigenvalue for A.
   (b) Prove that A is diagonalizable over $\mathbf{C}$.
   (c) Prove that A is not diagonalizable over $\mathbf{R}$.

3. (10 points each) Let V be a finite dimensional vector space over a field F, and W_1 and W_2 subspaces of V.
   (a) Let $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$. Prove that $W_1 + W_2$ is a subspace of V. (result from course)
   (b) Prove $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. (result from course)
   (c) Suppose further that $\dim(W_1) = \dim(W_2)$. Prove that there is a linear transformation $T\colon V \to V$ such that the following conditions all hold:
      i. $R(T) = W_1 + W_2$
      ii. $T(W_1) = W_2$ and $T(W_2) = W_1$
      iii. $T(v) = -v$ for all $v \in W_1 \cap W_2$

4. (5 points each) Suppose that V is a finite dimensional inner product space, W is a subspace, and $v \in V$.
   (a) Prove that there exists $w_0 \in W$ such that $v - w_0 \in W^\perp$. (result from course)
   (b) Prove that for all $w \in W$, $||v - w|| \geq ||v - w_0||$. (result from course)
   (c) Let $V = C[0, 1]$, the vector space of continuous functions on $[0, 1]$ and let

   $$\langle f(x), g(x) \rangle = \int_0^1 x f(x) g(x)\, dx$$

   be an inner product on V. Let $f_1(x) = \sqrt{2}$ and $f_2(x) = 6x - 4$. You may use that each of these functions has length 1 for the given inner product.
   Find the element w_0 from part (a) where W is the subspace of V spanned by $f_1(x)$ and $f_2(x)$ and $v = x^3$.

5. Let V be a finite dimensional inner product space over F.

   (a) (3 points) Define “normal operator”.

   (b) (5 points) Prove that if $[T]_\beta$ is diagonal for an orthonormal basis β of V, then T is normal. (result from course)

   (c) (12 points) Prove that if T is a normal linear operator on V such that the characteristic polynomial of T splits over F, then there exists an orthonormal basis β of V such that $[T]_\beta$ is diagonal. (result from course)

6. (5 points each) Let V be a finite dimensional vector space over a field F. A linear operator T on V is nilpotent if $T^j = T_0$ for some $j > 0$ (here T_0 denotes the zero-operator). A matrix A is nilpotent if the associated linear transformation, L_A, is nilpotent.

   (a) Prove that if T is nilpotent, then it has a Jordan canonical form over F.

   (b) Determine the number of similarity classes of nilpotent 5×5 matrices, and prove your answer is correct.

7. (10 points each) Let K be a field, F an extension field of K, $\alpha \in F$, $f(x) \in K[x]$ an irreducible polynomial such that $f(\alpha) = 0$.

   (a) Prove that if $g(x) \in K[x]$ and $g(\alpha) = 0$, then $f(x)h(x) = g(x)$ for some $h(x) \in K[x]$. (result from course)

   (b) Suppose p is a prime, n is a positive integer, and let $\mathbf{F}_p$ denote the field with p elements. Prove that if $x^{p^n} - x$ is factored as a product of irreducibles over $\mathbf{F}_p$, all of the factors are distinct.

   (c) Suppose $f(x) \in \mathbf{F}_p[x]$ is an irreducible polynomial of degree n. Prove that

$$f(x) \mid x^{p^n} - x\,.$$

   (d) Prove that if m is the degree of an irreducible factor of $x^{p^n} - x$, then $m \mid n$.

8. (5 points each) Let $F = \mathbf{Q}[t]/\langle t^3 - t - 2\rangle$.

   (a) Prove that F is a field. You may apply any theorems from the course in proving this, but give full statements of any theorems you do use.

   (b) Give a basis for F as a $\mathbf{Q}$-vector space. Again, carefully state any general theorems being used to know that the set you give is a basis.

   (c) Recall that if $z \in F$, then $T_z \colon F \to F$ given by $T_z(v) = zv$ is a linear transformation, as vector spaces over $\mathbf{Q}$. Using your basis from part (b), compute $[T_z]_\beta$.