Homework 4 for MAT 494 Due on Th. Nov. 9, 2006

(1) Consider the following simple SI model:

$$\begin{cases} x'(t) &= x + y - (1-\theta)y - x(x+y) - \delta x - s\dfrac{xy}{x+y} \equiv F(x,y), x(0) = x_0 > 0 \\ y'(t) &= -(\delta + r)y - y(x+y) + s\dfrac{xy}{x+y} \equiv G(x,y), y(0) = y_0 > 0 \end{cases} \quad (1)$$

where s, δ, r are positive constants and $\theta \in [0,1]$. To remove the nonsmoothness at the origin, we carry out the blow-up transformation $(x,y) \to (u,y)$ where $u = x/y$ in system (1). This transforms it into the following system:

$$\begin{aligned} u'(t) &= (1 + r - s)u + \theta, \\ y'(t) &= \varphi(u)y - (u+1)y^2, \end{aligned} \quad (2)$$

where

$$\varphi(u) = s - r - \delta - \frac{s}{u+1}, \quad (3)$$

Show the following statements are true.

i): Let $(x(t), y(t))$ be the solution of (1), starting at $(x_0, y_0) \in \mathbf{R}^2_+$. Then there is a $t_0 > 0$ such that for $t > t_0$, we have $(x(t), y(t)) \in \Omega = \{(x,y) \in \mathbf{R}^2_+ | x + y < 1\}$.

ii): The equilibrium $(1-\delta, 0)$ is globally attractive if either $s - r < 1$ and $\theta \in [0,1]$, or $s - r = 1$ and $\theta \in (0,1]$ hold.

iii): The equilibrium $(0,0)$ is globally attractive if $0 \le \theta \le 1 < s - r$ and $\varphi(u^*) \le 0$ where $u^* = \dfrac{\theta}{s-r-1}$.

(Reference: T.-W. Hwang and Y. Kuang: Host extinction dynamics in a simple parasite-host interaction model, Math. Biosc. and Eng., **2**, 743-751, (2005))

(2) Consider

$$\begin{aligned} S' &= -SI + 2(4 - S - I), \\ I' &= SI - 2I. \end{aligned}$$

a): Show that solutions with positive initial data are positive and bounded and

$$\limsup_{t\to\infty} (S(t) + I(t)) \le 4.$$

b): Show that the region $\Delta = \{(S,I) : 0 < S + I < 4, S > 0, I > 0\}$ is invariant.

c): Show that this SI epidemic model has no limit cycle in the positive cone and the positive steady state $(2,1)$ is globally asymptotically stable with respect to positive initial data. (Hint: use Dulac criterion).

(3) Consider the following food chain

$$\begin{aligned} x' &= x(5 - 2x - y), \\ y' &= y(-1 + 3x - z - y), \\ z' &= z(-1 + y). \end{aligned}$$

a): Find the local stability of the positive steady state of the food chain by the Routh–Hurwitz criteria.

b): Show that the positive steady state is globally stable.

(4): Exercise 21, page 206.
(5): Exercise 24, page 208.
(6): Exercise 22, page 261.
(7): Exercise 30, page 264.
(8): Exercise 33, page 266.
(9): Exercise 35, page 267.